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A SIMULATION ANALYSIS OF SOJOURN TIMES IN A JACKSON NETWORK.(U)
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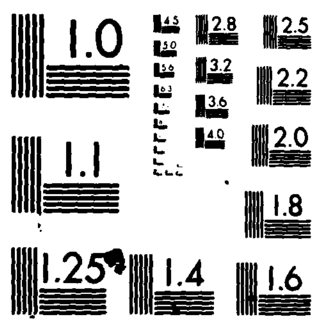

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Virginia Polytechnic Institute
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A SIMULATION ANALYSIS OF

SOJOURN TIMESIIN A JACKSON NETWORK

by

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TECH REPORT
VTR 8016
December 1980

This research was supported jointly by NSF Grant ENG77-22757 and by the Office of Naval Research Contract N00014-77-C-0743 (NR042-296).

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: Technical Report VTR 8016	2. GOVT ACCESSION NO.: AD-A093 797
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): (6) A Simulation Analysis of Sojourn Times in a Jackson Network.	5. TYPE OF REPORT & PERIOD COVERED: (9) Technical Report,
	6. PERFORMING ORG. REPORT NUMBER: VTR 8016
7. AUTHOR(s): (10) Peter C. Kiessler	8. CONTRACT OR GRANT NUMBER(s): (15) N00014-77-C-0743, NSF-ENG77-22757
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Department of Industrial Engr. & Operations Res. Virginia Polytechnic Inst. & State University Blacksburg, Virginia 24061	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: NR042-296 (12) 124
11. CONTROLLING OFFICE NAME AND ADDRESS: Director, Statistics and Probability Program Mathematical & Information Sciences Division 800 N. Quincy St., Arlington, VA 22217	12. REPORT DATE: (11) Dec~~ember 1~~980
	13. NUMBER OF PAGES: 123
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): (14) VTR-80-16	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Queueing Networks   Statistical Analysis

Sojourn Times

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This thesis analyzes the total sojourn time distribution of a three queue acyclic Jackson network with the following characteristics. The network consists of three single server queues, Q_1, Q_2, and Q_3, each having exponential service times. There is a Poisson exogenous arrival process to Q_1. Customers departing Q_1 go immediately to Q_2, w.p. p, or to Q_3, w.p. 1 - p. Customers departing Q_2 go

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immediately to Q_3 and those departing Q_3 leave the network. For different parameter settings, the network is simulated and the sojourn times at each of the queues are recorded. Due to the structure of the network, a simple simulation which correctly models the network can be constructed. It is known from Simon and Foley (1979), that the sojourn times of a customer in Q_1 and Q_3 are dependent. It is shown, that, except for extremely large sample sizes, 5000, that the correlation between the sojourn times in Q_1 and Q_3 is not significant. However, in the case of 5000 observations, this correlation is shown to be significantly greater than zero for certain parameter settings. Finally, the sample total sojourn time distribution is compared to one assuming independence of the sojourn times at each of the queues. It is shown that the sample distribution and the total sojourn time distribution assuming independence are not significantly different, except for $p = 0$.

This report is an interim report of on-going research. It may be amended, corrected or withdrawn, if called for, at the discretion of the author.

ACKNOWLEDGEMENTS

I would like to thank Professor Allen Soyster for serving on the committee.

I am grateful to Ms. Paula Myers for doing her typical outstanding job typing the manuscript. Her professionalism and pride in her work is present throughout the thesis. I cannot thank her enough.

I would like to thank Professor Robert D. Foley, who from the conceptualization of this work helped so much. His bright ideas and sometimes harsh but always honest criticism helped to make this document possible.

Finally, I would like to thank Professor Ralph L. Disney. However, to give him his due thanks would be impossible. He provided the funds to support this research through the grant, Office of Naval Research Contract N00014-77-C-0743 (NR042-296). He is also a fine teacher. He not only teaches probability and stochastic processes but also a sense of professionalism and dedication. He turned a $4.20 an hour press operator into, hopefully, a capable scientist. For this I will always be grateful.

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INTRODUCTION

1.1. Queueing Networks.

1.1.1. Single Queueing System.

In a queue one has an entity capable of performing service - a service system. The service is provided to a stream of customers. The service system has m servers if the system can service a maximum of m customers simultaneously. The time to service a given customer is a random variable. The sequence of service times is called the service process. The demand process is usually specified by the length of the interval between consecutive arrivals. Each interval is a random variable and the sequence of intervals is called the arrival process. The described system is a single queueing system and is referred to as a queue (figure 1.1).

Customers that arrive for service when the service system is busy (I.e., all m servers are busy.) may either wait their turn to be served, depart immediately, or wait some amount of time then depart. We assume that every arriving customer waits until served. (I.e., there are no early departures.) The maximum number of customers that are allowed to wait for service is the queue capacity. We assume the queue capacity is infinite. The order in which customers are served is the queue discipline. While many disciplines can be and have been studied we will assume that customers are served in the order of their arrival. Such a discipline is called First-Come-First-Served (FCFS).

We assume that the intervals between consecutive arrivals are

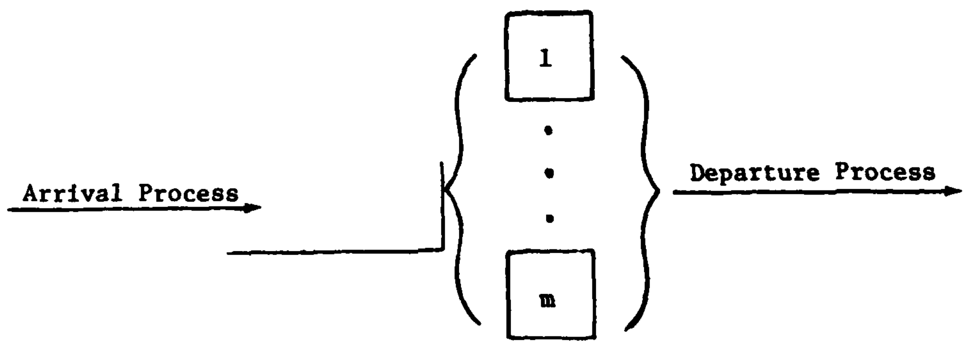


Figure 1.1. A Single Queue with m Servers

mutually independent and identically distributed, a renewal process. Such a process is called a GI process. A special case of the GI process is a Poisson process, designated as an M process, in which the intervals have an exponential probability distribution. The mean time between arrivals is designated as λ^{-1}.

The service times of the customers are mutually independent and identically distributed random variables. Such a process is denoted by GI. A special case of the GI process is designated as an M process, in which the service times are exponentially distributed random variables. The mean service time for each of the m servers is usually designated as μ^{-1}.

GI/GI/m will denote a queue when the first GI implies a renewal arrival process, the second GI a renewal service process, and the m the number of identical servers. These queues will have FCFS queue discipline and infinite queue capacity. Thus, M/M/m denotes a queue with a Poisson arrival process with mean interarrival time λ^{-1}, a service process with exponential distribution with mean μ^{-1}, and m identical servers.

1.1.2. Queueing Networks.

A queueing network is an aggregation of queues. For a survey on queueing networks see Disney (1975). The network's queues are interconnected by arcs on which customers travel. Customers that arrive from outside the network are called exogenous arrivals. Customers which travel from a queue to itself are called feedback customers. Customers which leave a network from a queue are called departures. All customers

entering a queue (i.e., the exogenous arrivals plus the arrivals from other queues) are called inputs. All customers leaving a queue are called outputs.

1.2. Sojourn Times in Queueing Networks.

For a single queue a customer's sojourn time consists of the time he spends waiting for service plus the time he spends receiving service. For the M/M/1 queue, with an arrival process with mean λ^{-1} and a service process with mean μ^{-1}, the sojourn time, S, of a customer has the distribution

$$P\{S \le t\} = 1 - e^{-(\mu-\lambda)t}$$

when the queue is in equilibrium, Kleinrock {(1973), pg. 202}.

We consider a queueing network with N queues, $Q_1,\cdots,Q_N$. A route ℓ is a sequence of queues a customer visits. We let $L = \{\ell_1,\ell_2,\cdots\}$ be the sets of all possible routes. Suppose a customer takes the route $\ell = \{Q_{i_1},\cdots,Q_{i_{m(\ell)}}\}$. Let S_{i_k}, $k = 1,\cdots,m(\ell)$, be the customer's sojourn time at Q_{i_k}. Let T_{i_j}, $j = 1,\cdots,m(\ell)-1$, be the customer's transition time from Q_{i_j} to $Q_{i_{j+1}}$. Then the customer's total sojourn time is given by

$$S = \sum_{k=1}^{m(\ell)} S_{i_k} + \sum_{j=1}^{m(\ell)-1} T_{i_j}. \tag{2.1}$$

The conditional distribution of the customer's total sojourn time given he takes route ℓ is given by

$$P\{S \le t|\ell\} = P\{\sum_{k=1}^{m(\ell)} S_{i_k} + \sum_{j=1}^{m(\ell)-1} T_{i_j} \le t|\ell\} \tag{2.2}$$

provided $p(\ell) > 0$ where $p(\ell)$ is the probability of the customer taking route ℓ. Multiplying (2.2) by $p(\ell)$ and summing over all $\ell \in L$ such that $p(\ell) > 0$ yields

$$P\{S \le t\} = \sum_{\substack{\ell \in L \\ p(\ell)>0}} P\{\sum_{k=1}^{m(\ell)} S_{i_k} + \sum_{j=1}^{m(\ell)-1} T_{i_j} \le t|\ell\}p(\ell) \tag{2.3}$$

the total sojourn time distribution.

1.3. <u>The Three Queue Network</u>.

We consider a specific network which consists of three queues, Q_1,Q_2,Q_3, figure 3.1. The exogenous arrivals to Q_1 form a Poisson process with mean λ^{-1}. For k = 1,2,3, Q_k consists of a single server who services customers according to an exponential distribution with mean μ_k^{-1}. Outputs from Q_1 go immediately to Q_2 with probability p or immediately to Q_3 with probability $(1-p)$. Outputs from Q_2 go immediately to Q_3 with probability 1. All outputs from Q_3 are departures.

The following two observations concern the total sojourn time distribution in the three queue network. First, transitions from one queue to another are instantaneous. Second, there are only two routes a customer may take. If ℓ_1 is the route Q_1,Q_2,Q_3 and ℓ_2 is the route Q_1,Q_3, then the total sojourn time, S, is given by

$$S = \begin{cases} S_1 + S_2 + S_3 & \text{with probability } p \\ S_1 + S_3 & \text{with probability } (1-p) \end{cases} \tag{3.1}$$

where S_i is the sojourn time at Q_i. From equation (2.1) the conditional

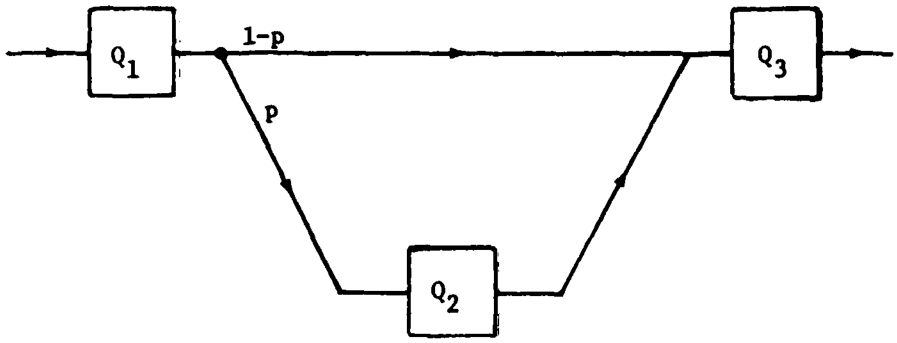


Figure 3.1. The Three Queue Network

sojourn time distributions are

$$P\{S \le t | \ell_1\} = P\{S_1 + S_2 + S_3 \le t | \ell_1\}$$

and

$$P\{S \le t | \ell_2\} = P\{S_1 + S_3 \le t | \ell_2\}.$$

Then since $p(\ell_1) = p$ and $p(\ell_2) = 1 - p$, the total sojourn time distribution is

$$P\{S \le t\} = pP\{S_1 + S_2 + S_3 \le t | \ell_1\} + (1 - p)P\{S_1 + S_3 \le t | \ell_2\}. \quad (3.2)$$

The difficulty with evaluating equation (3.2) is that given that a customer takes the route ℓ_1, his sojourn times in Q_1 and Q_3 are not independent. This will be discussed in detail in section (2.5). It is this dependence which makes determining the total sojourn time distribution difficult.

1.4. Purpose.

This thesis is concerned with a simulation study of the three queue network. The dependence of a customer's sojourn times in Q_1 and Q_3 given he takes the route ℓ_1, will be analyzed.

1.5. Summary of Results.

It is shown that for sample sizes of 1000 the correlation between S_1 and S_3 is not significantly different from zero. Further, the sample distribution of the total sojourn time, taken from the 1000 observation, is not significantly different from a distribution assuming S_1,S_2, and S_3 are mutually independent. Thus for applied modeling

purposes one can assume that S_1 and S_3 are uncorrelated and independent.

1.6. Organization.

This thesis consists of six chapters and two appendices. Chapter 2 consists of sojourn time and queue length results in Jackson networks. The major result, with respect to this thesis, is the theorem of Simon and Foley section (2.5) in which S_1 and S_3 are shown to be dependent given the customer takes route ℓ_1. In Chapter 3 the three queue network is put into the Markov framework. Once put into that structure, many of the properties of the network become apparent and thus, can be used for setting up the simulation.

Theorem (3.5) is of importance to the simulation study. Let $Y = \{Y_t;\ t \geq 0\}$ be the queue length process at any $t \geq 0$ and $Z = \{Z_n;\ n \in \mathbb{N}\}$ be the queue length process just before the nth arrival to Q_1. For computing the equilibrium sojourn time of a customer the Z process must be in equilibrium. Theorem (3.5) shows that if Y has an equilibrium distribution at time t, then the next arrival to Q_1 does not see the equilibrium distribution of Z. Thus, in simulating the network to find a customer's equilibrium sojourn time one must generate the equilibrium distribution of Z and then immediately add the tagged customer. Further, if the initial distribution is generated according to the equilibrium distribution of the Y process and the network is simulated until the next arrival to Q_1, then this customer does not see the equilibrium distribution of the Z process. Thus, this customer's sojourn time is not the sojourn time of a customer who finds the network in equilibrium.

In Chapter 4, the method of the simulation is explained. That chapter explains the initialization of the simulation and the next event generator. Chapter 5 contains the analysis of the simulation. Two basic tests are done. First, a test is made to determine whether the correlation between S_1 and S_3 is significant. The second test determines whether the total sojourn time distribution is statistically different from a distribution assuming S_1, S_2, and S_3 are mutually independent. Chapter 6 contains the conclusions.

Appendix A1 contains flowcharts, a source listing of the program, and discriptions of the programs. Appendix A2 contains listings of the output. These listings consist of expected values and variances of sojourn times, correlation coefficients, and plots of the distributions.

Chapters are assigned an Arabic number. Each chapter is divided in sections and sections, when needed, into subsections. Sections are labeled by two Arabic numbers, one for the chapter and one for the section. Thus, the third section of Chapter 2 would be labeled 2.3. If this section has subsections, the second subsection would be labeled 2.3.2. Theorems, definitions, equations, tables, and figures within a section are labeled n_1, n_2, where n_1 and n_2 are Arabic numbers. Inside a chapter, the chapter number is suppressed. Thus, one would refer to Theorem (2.1) from inside Chapter 3, but Theorem (3.2.1) from outside Chapter 3.

CHAPTER 2

QUEUE LENGTH AND SOJOURN TIME RESULTS IN JACKSON NETWORKS

2.1. Jackson Networks.

A Jackson network consists of N queues, $Q_1,\cdots,Q_N$. For each n, $n = 1,\cdots,N$, Q_n consists of m_n identical servers. Servers at Q_n service customers according to an exponential distribution with mean μ_n^{-1}. The queue discipline at Q_n is FCFS and its queue capacity is infinite. Exogenous arrivals to Q_n form a Poisson process with mean λ_n^{-1}. Upon completing service at Q_n the departing customer goes instantaneously to Q_k, $k = 1,\cdots,N$, with probability Θ_{nk} or leaves the system, never to return, with probability $p_n = 1 - \sum_{k=1}^{N} \Theta_{nk}$.

The matrix Θ with elements Θ_{jk}, $j,k = 1,\cdots,N$, is substochastic since for each $j, j = 1,\cdots,N$,

$$\sum_{k=1}^{N} \Theta_{jk} \leq 1.$$

Form a matrix $\hat{\Theta}$ by appending to Θ an extra row and an extra column as follows. The element in column $N+1$ for row j, $j = 1,\cdots,N$, is p_j. The elements in row $N+1$ are given by

$$\hat{\Theta}_{N+1j} = \begin{cases} 0 & \text{for } j = 1,\cdots,N \\ 1 & \text{for } j = N+1. \end{cases}$$

The matrix,

$$\hat{\Theta} = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \cdots & \Theta_{1N} & p_1 \\ \Theta_{21} & \Theta_{22} & \cdots & \Theta_{2N} & p_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta_{n1} & \Theta_{n2} & \cdots & \Theta_{NN} & p_N \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

We will denote by $\underset{\sim}{\lambda}$ the row vector $(\lambda_1,\cdots,\lambda_N)$, by $\underset{\sim}{m}$ the row vector $(m_1,\cdots,m_N)$, and by $\underset{\sim}{\mu}$ the row vector $(\mu_1,\cdots,\mu_m)$. $\pi_{k_1,\cdots,k_N}$ is the probability that initially, $t=0$, there are k_1 customers in $Q_1,\cdots,k_N$ customers in Q_N. The quintuplet N, $\underset{\sim}{\lambda}$, $\underset{\sim}{m}$, $\underset{\sim}{\mu}$, Θ plus the initial distribution $\pi_{k_1,\cdots,k_N}$ completely specifies the Jackson network. We will denote a Jackson network specification by $JN = (N, \underset{\sim}{\lambda}, \underset{\sim}{m}, \underset{\sim}{\mu}, \Theta)$ with initial distribution $\pi_{k_1,\cdots,k_N}$. When the initial distribution is not relevant it will be deleted from the specification.

<u>Definition</u> 1.1. A Jackson network specified by $JN = (N, \lambda, \underset{\sim}{m}, \underset{\sim}{\mu}, \Theta)$ is open if for every $j \in \{1,\cdots,N\}$ the probability of never departing the network starting from j is zero.

Let $JN = (N, \underset{\sim}{\lambda}, \underset{\sim}{m}, \underset{\sim}{\mu}, \Theta)$ specify a Jackson network. If the input process to Q_n, $n=1,\cdots,N$, has rate Γ_n, then when the network is in equilibrium, (I.e., the joint queue length distribution is time invariant.)

$$\Gamma_n = \lambda_n + \sum_{k=1}^{N} \Theta_{kn}\Gamma_k. \tag{1.2}$$

This equation is known as the traffic equation. An intuitive explanation for this equation is as follows. In equilibrium it is known that the

rate of flow into a queue equals the rate of flow out of a queue. Thus, for $k=1,\cdots,N$, the rate of flow out of Q_k is Γ_k. So the flow into Q_n becomes the portion of flow out of Q_k, $k=1,\cdots,N$, which goes to Q_n, $\Theta_{kn}\Gamma_k$, plus the rate, λ_n, of Q_n's exogenous arrival process yielding equation (1.2). In matrix form equation (1.2) becomes

$$\underset{\sim}{\Gamma} = \underset{\sim}{\lambda} + \underset{\sim}{\Gamma}\Theta \tag{1.3}$$

where $\underset{\sim}{\Gamma} = (\Gamma_1,\cdots,\Gamma_N)$.

Let $JN = (N, \underset{\sim}{\lambda}, \underset{\sim}{m}, \underset{\sim}{\mu}, \Theta)$ specify an open Jackson network. Melamed (1976) shows that equation (1.3) has an unique solution given by

$$\underset{\sim}{\Gamma} = \underset{\sim}{\lambda} \sum_{n=0}^{\infty} \Theta^n \tag{1.4}$$

where Θ^o is the idenity matrix.

Let $JN = (N, \underset{\sim}{\lambda}, \underset{\sim}{m}, \underset{\sim}{\mu}, \Theta)$ specify an open Jackson network. Jackson (1957) derived the equilibrium distribution for the queue length vector $\underset{\sim}{k} = (k_1,\cdots,k_N)$ where k_i, $i=1,\cdots,N$, is the number of customers at Q_i. Define $P_k^{(n)}$ $(n=1,\cdots,N,\ k=0,1,\cdots)$ by the following equations

$$P_k^{(n)} = \begin{cases} P_o^{(n)} (\Gamma_n/\mu_n)^k/k! & k=0,\cdots,m_n \\ P_o^{(n)} (\Gamma_n/\mu_n)^k/m_n!\,m_n^{k-m_n} & k=m_n,m_n+1,\cdots \end{cases} \tag{1.5}$$

where $P_o^{(n)}$ can be determined by the equations $\sum_{k=0}^{\infty} P_k^{(n)} = 1$. Jackson shows that the equilibrium distribution of the queue length vector in an open Jackson network is given by

$$P_{\underset{\sim}{k}} = P_{k_1}^{(1)} \cdots P_{k_N}^{(N)}$$

provided that $\Gamma_n < m_n\mu_n$ for $n=1,\cdots,N$.

2.2. Acyclic Jackson Networks.

A Jackson network is said to be acyclic if it has the additional property that once a customer departs from Q_n, $n = 1,\cdots,N$, he may never return to Q_n. The following result is concerned with the switching matrix, Θ, of an acyclic Jackson network.

Theorem 2.1. Let $JN = (N, \underset{\sim}{\lambda}, \underset{\sim}{m}, \underset{\sim}{\mu}, \Theta)$ specify a Jackson network. Then JN specifies an acyclic Jackson network iff Θ can be put into upper triangular form.

Proof. ($\Longleftarrow$) Suppose $\bar{\Theta}$ is a switching matrix of an acyclic Jackson network. Then there is a row i_1 consisting of all zeros. For if not, for each $i \in \{1,\cdots,N\}$ there would exist an $n \in \{1,2,\cdots\}$ such that $\bar{\Theta}^n_{ii} > 0$. Thus $\bar{\Theta}$ is not the switching matrix of an acyclic Jackson network. Similarly, there exists a column j of $\bar{\Theta}$ consisting of all zeros.

Form a matrix $\bar{\Theta}_1$ by deleting the i_1^{th} row and i_1^{th} column of the matrix $\bar{\Theta}$. Then $\bar{\Theta}_1$ has a row i_2 consisting of all zeros by the same reason as above. Continuing in this manner we get matrices $\bar{\Theta}_k$ with row i_{k+1} consisting of all zeros. Let Θ be the matrix formed by the row and column ordering $i_N,\cdots,i_1$. Then Θ is in upper triangular form.

($\Longrightarrow$) Suppose Θ is in upper triangular form. Then for each $i \in \{1,\cdots,N\}$, $\Theta^n_{ii} = 0$ for all $n \geq 1$. □

Using this result it follows that an acyclic Jackson network is an open network.

Theorem 2.2. Let $JN = (N, \underset{\sim}{\lambda}, \underset{\sim}{m}, \underset{\sim}{\mu}, \Theta)$ specify an acyclic Jackson network. Then JN specifies an open network.

Proof. Consider the matrix $\hat{\Theta}$ formed from the matrix Θ as in equation (1.1). Since Θ is in upper triangular form it follows that for

any i, i ϵ $\{1,\cdots,N\}$, that

$$\hat{\theta}_{1N+1}^{N+1-i} = 1. \quad \square$$

The three queue network, section (1.3), is an acyclic Jackson network with switching matrix

$$\Theta = \begin{bmatrix} 0 & p & q \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} . \tag{2.1}$$

Since the three queue network is an open network, it follows that in equilibrium the solution to the traffic equation (1.3) is given by equation (1.4). Thus, it follows that

$$\underset{\sim}{\Gamma} = \underset{\sim}{\lambda} \sum_{n=0}^{\infty} \Theta^n$$

$$= (\lambda,0,0) \left(\sum_{n=0}^{\infty} \begin{pmatrix} 0 & p & q \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^n \right)$$

$$= (\lambda,0,0) \begin{pmatrix} 1 & p & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= (\lambda,p\lambda,\lambda). \tag{2.2}$$

For the three queue network, the equilibrium queue length vector is given by equation (1.5) as

$$P_{k_1,k_2,k_3} = (1-\lambda/\mu_1)(\lambda/\mu_1)^{k_1}(1-p\lambda/\mu_2)(p\lambda/\mu_2)^{k_2}(1-\lambda/\mu_3)(\lambda/\mu_3)^{k_3} \tag{2.3}$$

for $k_1,k_2,k_3 = 0,1,\cdots$, provided λ/μ_1, $p\lambda/\mu_2$, $\lambda/\mu_3 < 1$.

2.3. <u>Sojourn Times in Acyclic Jackson Networks</u>.

Let $JN = (N, \underset{\sim}{\lambda}, \underset{\sim}{m}, \underset{\sim}{\mu}, \Theta)$ specify an acyclic Jackson network in equilibrium. Suppose a customer, c, takes the route $\ell = \{\ell_1,\cdots,\ell_m\}$. The probability that c takes the route ℓ is

$$p(\ell) = \frac{\lambda_{\ell_1}}{\sum_{i=1}^{N} \lambda_{\ell_i}} \; \theta_{\ell_1\ell_2}\cdots\theta_{\ell_{m-1}\ell_m} p_{\ell_m}. \tag{3.1}$$

So by equation (1.3.3), c's total sojourn time distribution is given by

$$P\{S \leq t\} = \sum_{\ell \in L} P\{S_{\ell_1} + \cdots + S_{\ell_m} \leq t \,|\, \ell\}\left(\frac{\lambda_{\ell_1}}{\sum_{i=1}^{N} \lambda_{\ell_i}} \; \theta_{\ell_1\ell_2}\cdots\theta_{\ell_{m-1}\ell_m} p_{\ell_m}\right). \tag{3.2}$$

If for each route $\ell = \{\ell_1,\cdots,\ell_m\} \in L$, the S_{ℓ_i}, $i = 1,\cdots,m$, are mutually independent random variables then the total sojourn time distribution is given by

$$P\{S \leq t\} = \sum_{\ell \in L} (F_{\ell_1} * \cdots * F_{\ell_m})\left(\frac{\lambda_{\ell_1}}{\sum_{i=1}^{N} \lambda_{\ell_i}} \; \theta_{\ell_1\ell_2}\cdots\theta_{\ell_{m-1}\ell_m} p_{\ell_m}\right) \tag{3.3}$$

where F_i is the distribution of S_i, $i = \{1,\cdots,N\}$, and $F * G$ denotes the convolution of F and G.

Let $JN = (N, \underset{\sim}{\lambda}, \underset{\sim}{1}, \underset{\sim}{\mu}, \Theta)$ specify an acyclic Jackson network in equilibrium where $\underset{\sim}{1} = (1,\cdots,1)$. Beutler and Melamed (1979) show that the traffic on the arc connecting Q_i, and Q_j, $i,j = 1,\cdots,N$, forms a Poisson process with mean $(\Gamma_i\theta_{ij})^{-1}$, and is independent of all other

traffic in the network. Thus, the input process to Q_j, $j = 1,\cdots,N$, is the superposition of mutually independent Poisson processes. Therefore, the input process to Q_j is a Poisson process with mean

$\Gamma_j^{-1} = (\lambda_j + \sum_{i=1}^{N} \Gamma_i \Theta_{ij})^{-1}$, (cf. Çinlar (1975) pg. 87). Further, the input process to Q_j, $j = 1,\cdots,N$, is independent of its service process. So it follows that Q_j, $j = 1,\cdots,N$, is a M/M/1 queue. It follows from equation (1.3.1) that Q_j's sojourn time distribution is exponential with mean $(\mu_j - \Gamma_j)^{-1}$. Thus, if for each route $\ell = \{\ell_1,\cdots,\ell_m\}$, the S_{ℓ_i}, $i = 1,\cdots,m$, were mutually independent, the total sojourn time distribution would be the sum of convolutions of exponential distributions.

In general the S_{ℓ_i} are not mutually independent random variables. Thus, equation (3.3) does not hold. However, there are some types of networks in which equation (3.3) holds.

2.4. Sojourn Times in Tandem Queues and Tree-like Networks.

$JN = (N, \underset{\sim}{\lambda}, \underset{\sim}{m}, \underset{\sim}{\mu}, \Theta)$ specifies a Jackson network of tandem queues, figure 4.1, if

$$\Theta = \begin{bmatrix} 0 & 1 & & & & \text{O} \\ & \cdot & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & 1 \\ & \text{O} & & & & 0 \end{bmatrix}$$

and $\underset{\sim}{\lambda} = (\lambda,0,\cdots,0)$. JN specifies a tree-like network, figure 4.2, if for any $i,j \in \{1,\cdots,N\}$, there is at most one sequence $\{i,k_1,\cdots,k_n,j\}$ such that $\Theta_{ik_1}\cdots\Theta_{k_nj} > 0$.

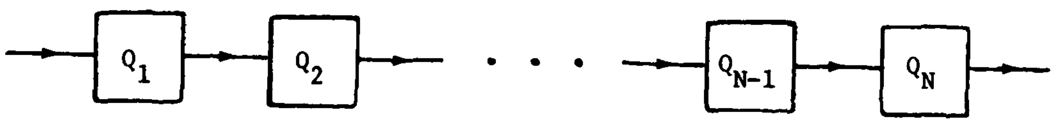


Figure 4.1. N Queues In Tandem

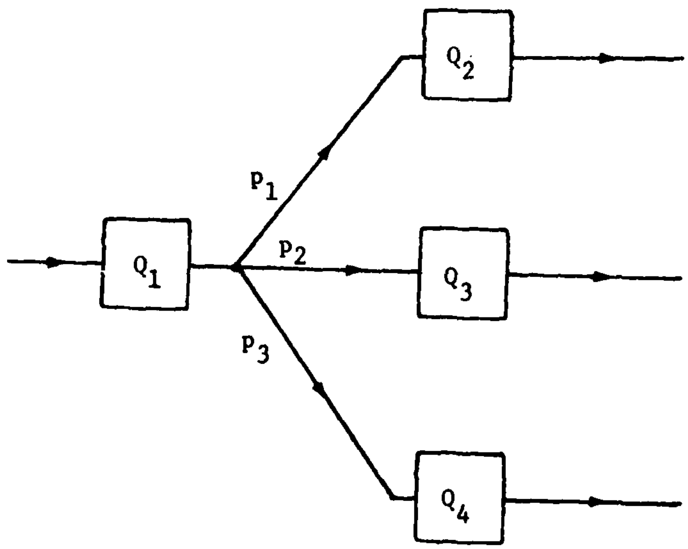


Figure 4.2. A Tree-like Network

Let $JN = (N, \underset{\sim}{\lambda}, \underset{\sim}{1}, \underset{\sim}{\mu}, \Theta)$ specify a Jackson network of tandem queues in equilibrium. Since the only route, ℓ, a customer, c, may take is $\ell = \{1,\cdots,N\}$, equation (1.2.3) becomes

$$P\{S \leq t\} = P\{S_1+\cdots+S_N \leq t\}. \tag{4.1}$$

Reich (1963) shows that the S_i, $i = 1,\cdots,N$, are mutually independent. Thus, from equation (3.3)

$$P\{S \leq t\} = (F_1*\cdots*F_N)(t)$$

where each of the F_i, $i = 1,\cdots,N$, are exponential distributions with mean $(\mu_i - \lambda)^{-1}$.

Melamed (1979) extends Reich's result to single server tree-like networks. Let $JN = (N, \underset{\sim}{\lambda}, \underset{\sim}{1}, \underset{\sim}{\mu}, \Theta)$ specify a tree-like Jackson network in equilibrium. Melamed shows that for each route, $\ell = \{\ell_1,\cdots,\ell_m\} \in L$, the S_{ℓ_i} are mutually independent and thus the total sojourn time distribution of a customer is given by equation (3.3).

So far we have examined Jackson networks of tandem queues and Jackson tree-like networks with single server queues. What happens if these networks have queues with multi-servers? Burke (1968) shows that for two multi-server queues in tandem, the sojourn times of a customer in the first queue and the second queue are independent. However, mutual independence of the sojourn times of a customer is not the case when one tries to extend the result to Jackson networks of multi-server tandem queues of three or more queues.

Burke (1969) constructs a network of three queues in tandem, see

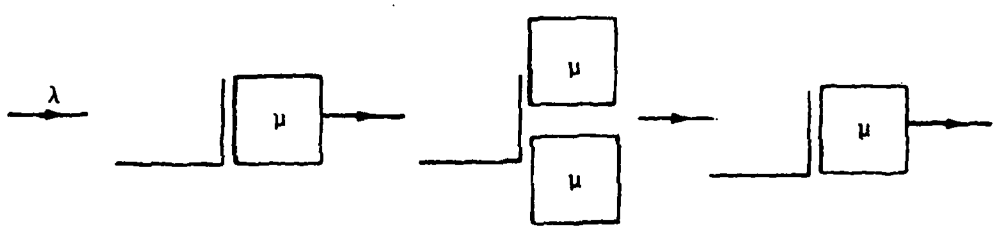


Figure 4.3. The Burke Network

figure 4.3, such that the sojourn times at the separate queues are not mutually independent. Let $JN = (N, \underset{\sim}{\lambda}, \underset{\sim}{m}, \underset{\sim}{\mu}, \Theta)$ specify a Jackson network of tandem queues in equilibrium with $N = 3$, $\underset{\sim}{m} = (1,2,1)$, and $\underset{\sim}{\mu} = (\mu,\mu,\mu)$. Let a customer, c, have a sojourn, s_1, in Q_1. Let s_1 be large enough as to guarantee the arrival of another customer to Q_1 before c's departure from Q_1. Burke shows that c's expected sojourn time in Q_3 conditioned on s_1 can be bounded below by $1/8\mu + 1/\mu$. However, the unconditioned expected sojourn time of a customer at Q_3 is given by $\lambda/\lambda+\mu \cdot 1/\mu + 1/\mu$. Thus, choosing $\lambda/\lambda+\mu < 1/8$ the conditional and unconditional sojourn times at Q_3 cannot be equal proving the dependence of sojourn times at Q_1 and Q_3.

2.5. Sojourn Times in the Three Queue Network.

Simon and Foley (1979) show that for their three queue network in equilibrium, section (2.2), the sojourn times of a customer in Q_1 and Q_3 are dependent given that the customer takes the route Q_1,Q_2,Q_3. Their result is shown in a manner similar to Burke (1969) discussed in the previous section.

Simon and Foley show that for any fixed $r > 0$, μ_1 and s can be chosen so that

$$E\{S_3 | S_1 = s\} > r.$$

The unconditioned expected sojourn time of a customer in Q_3 is $(\mu_3 - \lambda)^{-1}$. Choosing $r > (\mu_3 - \lambda)^{-1}$ one obtains

$$E\{S_3 | S_1 = s\} > E\, S_1$$

proving the dependence of S_1 and S_3.

The following is an intuitive explanation for the dependence of S_1 and S_3. In the three queue network a customer, c, may either go from Q_1 to Q_3 directly or from Q_1 to Q_2 then to Q_3. The number of customers, n_1, at c's departure time, t, from Q_1 depends on S_1. If c goes to Q_2 from Q_1, some of the n_1 customers may bypass c by going directly to Q_3. Thus, when c arrives at Q_3, the queue length there, n_3, is dependent on n_1. Then since S_3 is dependent on n_3, n_3 on n_1, and n_1 on S_1, S_3 is dependent on S_1.

2.6. Summary.

In section (2.1) we defined a Jackson network and discussed the switching matrix Θ. Further, the traffic equation $\underset{\sim}{\Gamma} = \underset{\sim}{\lambda} + \underset{\sim}{\Gamma}\Theta$ was defined and the solution to this equation for an open network was discussed. In section (2.2) we defined an acyclic network. We showed for acyclic networks Θ could always be put in upper triangular form. Thus, an acyclic network is an open network. In section (2.4) we discussed sojourn times in Jackson networks of tandem queues and Jackson tree-like networks. For the cases of networks of single server queues or two queues in tandem we had mutual independence of sojourn times. However, this result could not be extended to these networks in general. Finally, in section (2.5), Simon and Foley showed that for the three queue network we did not have independence of the sojourn times.

In the next chapter we discuss some properties of the three queue network's queue length process which are needed in the construction of the simulation.

CHAPTER 3

PROPERTIES OF THE QUEUE LENGTH PROCESS

3.0. Introduction.

In this chapter we will study four vector valued queue length processes, Y, X, $\hat{X}$, Z. The process $Y = \{Y_t: t \geq 0\}$ is the queue length (at each server) process at $t \in R$. The process $X = \{X_n: n = 0,1,2,\cdots\}$ is the queue length process at the time of the nth jump in the Y process. The process $\hat{X} = \{\hat{X}_n: n = 0,1,2,\cdots\}$ is the queue length process at the time of the nth jump in a Poisson process that is independent of Y. The process $Z = \{Z_n: n = 0,1,2,\cdots\}$ is the queue length process at the time of the nth arrival (T_n^a) to the network.

We will:

1. define the Y process and discuss a few of its properties (section 3.1);

2. define the X process, show that it is a Markov chain, find its one step transition probabilities and from these find the infinitesimal generator of Y (section 3.2);

3. define a process $\hat{X}$ that is useful in the study of the queue length process embedded at arrival times (i.e., Z). We show that $\hat{X}$ and Y have the same invariant distributions;

4. define the queue length process embedded at arrival times and show it has the same invariant distributions as $\hat{X}$ (and thus of Y);

5. show that if Y is in equilibrium at some time t_o then Z at the first arrival after t_o is not in equilibrium.

All of these properties are important to the simulation methods used

in Chapter 5 as we discuss in the summary to this chapter.

3.1. The Queue Length Process, Y, of the Three Queue Network.

Let JN = (N, $\underset{\sim}{\lambda}$, $\underset{\sim}{m}$, $\underset{\sim}{\mu}$, Θ) where:

(i) $N = 3$;

(ii) $\underset{\sim}{\lambda} = (\lambda, 0, 0)$;

(iii) $\underset{\sim}{m} = (1, 1, 1)$;

(iv) $\underset{\sim}{\mu} = (\mu_1, \mu_2, \mu_3)$;

(v)

$$\Theta = \begin{bmatrix} 0 & p & (1-p) \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} .$$

Then JN specifies the three queue network.

Let $Y = \{Y_t;\ t \geq 0\}$ be a stochastic process taking values in the state space $E = \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, where $\mathbb{N} = \{0,1,\cdots\}$. Assume that Y is a right continuous pure jump process with jumps of the following form. If $T \geq 0$ is the time of a jump in the process Y and if $Y_{T-} = (i_1, i_2, i_3)$, then Y_{T+} can take the values:

(i) $(i_1 + 1, i_2, i_3)$;

(ii) $(i_1 - 1, i_2 + 1, i_3)$ provided $i_1 \geq 1$;

(iii) $(i_1 - 1, i_2, i_3 + 1)$ provided $i_1 \geq 1$;

(iv) $(i_1, i_2 - 1, i_3 + 1)$ provided $i_2 \geq 1$;

(v) $(i_1, i_2, i_3 - 1)$ provided $i_3 \geq 1$.

The stochastic process Y defined above is the queue length process of JN. It can be shown from Jackson (1957) that Y is a irreducible recurrent homogeneous Markov process.

Suppose at time t, $Y_t = (i_1, i_2, i_3)$. Let T be the time from t until the next jump of the Y process. Then

$$P_{\underset{\sim}{i}}\{T > s\} = P_{\underset{\sim}{i}}\{A^T > s,\ D_1^T > s,\ D_2^T > s,\ D_3^T > s\}$$

where

$$P_{\underset{\sim}{i}}\{\cdot\} = P\{\cdot \mid Y_u;\ u < t,\ Y_t = \underset{\sim}{i}\};$$

A^T is the time from t until the next arrival to Q_1;

D_i^T is the time from t until the next departure from Q_i, i = 1,2,3.

Now,

$$P_{\underset{\sim}{i}}\{A^T > s,\ D_1^T > s,\ D_2^T > s,\ D_3^T > s\} = P_{\underset{\sim}{i}}\{A^T > s\}P_{\underset{\sim}{i}}\{D_1^T > s \mid A^T > s\}$$

$$P_{\underset{\sim}{i}}\{D_2^T > s \mid D_1^T > s,\ A^T > s\}P_{\underset{\sim}{i}}\{D_3^T > s \mid D_2^T > s,\ D_1^T > s,\ A^T > s\}.$$

First,

$$P_{\underset{\sim}{i}}\{A^T > s\} = e^{-\lambda s}$$

since the arrival process is Poisson with mean λ^{-1}. Second,

$$P_{\underset{\sim}{i}}\{D_1^T > s \mid A^T > s\} = \begin{cases} 1 & \text{if } i_1 = 0 \\ e^{-\mu_1 s} & \text{if } i_1 \geq 1; \end{cases} \tag{1.1}$$

$$P_{\underset{\sim}{i}}\{D_2^T > s \mid D_1^T > s,\ A^T > s\} = \begin{cases} 1 & \text{if } i_2 = 0 \\ e^{-\mu_2 s} & \text{if } i_2 \geq 1; \end{cases} \tag{1.2}$$

$$P_{\underset{\sim}{i}}\{D_3^T > s \mid D_2^T > s,\ D_1^T > s,\ A^T > s\} = \begin{cases} 1 & \text{if } i_3 = 0 \\ e^{-\mu_3 s} & \text{if } i_3 \geq 1. \end{cases} \tag{1.3}$$

Equations (1.1), (1.2), and (1.3) are explained as follows. If $i_k = 0$, $k = 1,2,3$, and the next arrival to Q_k is after $t + s$ then there cannot be a departure from Q_k until after $t + s$. If $i_k \geq 1$, then the remaining service time of the customer at Q_k has exponential distribution with mean μ_k^{-1}. Thus,

$$P_{\underset{\sim}{i}}\{T > s\} = e^{-(\lambda + \mu_1(i_1) + \mu_2(i_2) + \mu_3(i_3))s} \tag{1.4}$$

where

$$\mu_k(i_k) = \begin{cases} 0 & i_k = 0 \\ \mu_k & i_k \geq 1 \end{cases} \qquad \text{for } k = 1,2,3.$$

3.2. <u>The Queue Length Process X.</u>

It is shown below that the queue length process X, which is the process embedded at jump points of Y, is a Markov chain. Its one step transition probabilities are found, equation (2.1), which lead directly to the infinitesimal generator of the Y process, equation (2.2).

Let T_n be the time of the nth jump of the Markov process Y. It then follows that the process $X = \{X_n; n \in \mathbb{N}\} = \{Y(T_n); n \in \mathbb{N}\}$ is a Markov chain (cf. Çinlar (1975) pg. 247, 254). Now

$$P\{X_{n+1} = (i_1 + 1, i_2, i_3) | X_n = (i_1, i_2, i_3)\}$$

$$= P\{A < D_1, A < D_2, A < D_3 | X_n = (i_1, i_2, i_3)\}$$

where A is the next arrival to Q_1 after T_n and D_i is the next departure from Q_i, $i = 1,2,3$, after T_n.

$$P\{A < D_1, A < D_2, A < D_3 | X_n = (i_1, i_2, i_3)\}$$

$$= E\{P\{A < D_1, A < D_2, A < D_3 | X_n = (i_1, i_2, i_3), A = t\} | X_n = (i_1, i_2, i_3)\}$$

$$= \int_0^\infty e^{-(\mu_1(i_1) + \mu_2(i_2) + \mu_3(i_3))t} \lambda e^{-\lambda t} dt$$

$$= \frac{\lambda}{\gamma(\underset{\sim}{i})}$$

where $\gamma(\underset{\sim}{i}) = \lambda + \mu_1(i_1) + \mu_2(i_2) + \mu_3(i_3)$. In a similar manner the other transition probabilities of X are found as:

$$Q(\underset{\sim}{i},\underset{\sim}{j}) = \begin{cases} \dfrac{\lambda}{\gamma(\underset{\sim}{i})} & \underset{\sim}{j} = (i_1 + 1, i_2, i_3); \\[2ex] \dfrac{p\mu_1(i_1)}{\gamma(\underset{\sim}{i})} & \underset{\sim}{j} = (i_1 - 1, i_2 + 1, i_3); \\[2ex] \dfrac{(1-p)\mu_1(i_1)}{\gamma(\underset{\sim}{i})} & \underset{\sim}{j} = (i_1 - 1, i_2, i_3 + 1); \\[2ex] \dfrac{\mu_2(i_2)}{\gamma(\underset{\sim}{i})} & \underset{\sim}{j} = (i_1, i_2 - 1, i_3 + 1); \\[2ex] \dfrac{\mu_3(i_3)}{\gamma(\underset{\sim}{i})} & \underset{\sim}{j} = (i_1, i_2, i_3 - 1); \\[2ex] 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

From Çinlar (1975, pg. 254) the infinitesimal generator of Y is

$$A(\underset{\sim}{i},\underset{\sim}{j}) = \begin{cases} -\gamma(\underset{\sim}{i}) & \text{if } \underset{\sim}{i} = \underset{\sim}{j}, \\[2ex] \gamma(\underset{\sim}{i})Q(\underset{\sim}{i},\underset{\sim}{j}) & \text{if } \underset{\sim}{i} \neq \underset{\sim}{j}. \end{cases} \tag{2.2}$$

3.3. The Queue Length Process $\hat{X}$.

It is shown below that the queue length process, $\hat{X}$, embedded at the instants of jumps of a Poisson process independent of Y is a Markov chain with the same invariant distribution as Y.

Let $N^e = \{N_t^e;\ t \geq 0\}$ be a Poisson process with mean λ^{-1} independent of Y. Note that N^e is not the exogenous arrival process to Q_1. The following lemma shows that the process $(Y,N^e) = \{Y_t, N_t^e;\ t \geq 0\}$ is a Markov process on the state space $E \times \mathbb{N}$.

Lemma 3.1. The process $(Y,N^e) = \{Y_t, N_t^e;\ t \geq 0\}$ is a Markov process.

Proof. $P\{Y_{t+s}=i, N_{t+s}^e=j \mid Y_u, N_u^e;\ u \leq t\} = P\{Y_{t+s}=i, N_{t+s}^e - N_t^e = k \mid Y_u, N_u^e;\ u \leq t\}$

for $Y_t = \ell,\ N_t = h = k - j$

$$= P\{Y_{t+s}=i \mid Y_u;\ u \leq t\}\, P\{N_{t+s}^e - N_t^e = k \mid N_u;\ u \leq t\}$$

by the independence of Y and N^e

$$= P\{Y_{t+s}=i \mid Y_t=\ell\}\, P\{N_{t+s}^e - N_t^e = k \mid N_t^e = h\}$$

since Y is Markov and N^e Poisson

$$= P\{Y_s=i \mid Y_o=\ell\}\, P\{N_s^e - N_0^e = k \mid N_o^e = h\}$$

by the homogeneity of Y and N^e

$$= P\{Y_s=i, N_s^e=\ell \mid Y_o=k, N_o^e=h\}. \quad \square$$

Let T_n^e be the time of the nth jump of N^e. It is apparent that T_n^e is a stopping time of (Y,N^e). Let $\hat{X} = \{\hat{X}_n;\ n \in \mathbb{N}\} = \{Y(T_n^{e-});\ n \in \mathbb{N}\}$. Then the following theorem shows that $\hat{X}$ is a Markov chain.

<u>Theorem</u> 3.2. $\{\hat{X}_n;\ n \in \mathbb{N}\} = \{Y(T_n^{e}-);\ n \in \mathbb{N}\}$ is an homogenous, irreducible, recurrent Markov chain with transition matrix given by

$$P(\underset{\sim}{i},\underset{\sim}{j}) = \lambda U^{\lambda}(\underset{\sim}{i},\underset{\sim}{j}) = \int_0^{\infty} \lambda e^{-\lambda t} U^{\lambda}(\underset{\sim}{i},\underset{\sim}{j})dt$$

for $\underset{\sim}{i},\underset{\sim}{j} \in E$, where $\lambda U^{\lambda}(\underset{\sim}{i},\underset{\sim}{j}) = \lambda \int_0^{\infty} P_t(\underset{\sim}{i},\underset{\sim}{j})e^{-\lambda t}dt$ (Çinlar (1975) pg. 256). Further, $\hat{X}$ and Y have the same invariant distribution.

<u>Proof</u>. $P\{\hat{X}_{n+1} = \underset{\sim}{j} | \hat{X}_o, \cdots, \hat{X}_n = \underset{\sim}{i}\} = P\{Y(T_{n+1}^{e}-) = \underset{\sim}{j} | Y(T_o^{e}-), \cdots, Y(T_n^{e}-) = \underset{\sim}{i}\}$

$$= P\{Y(T_1^{e}-) = \underset{\sim}{j} | Y(T_o^{e}-) = \underset{\sim}{i}\}$$

by the strong Markov property and homogeneity of Y

$$= P\{\hat{X}_1 = \underset{\sim}{j} | \hat{X}_o = \underset{\sim}{i}\}.$$

Thus, $\hat{X}$ is a homogeneous Markov chain.

Since any state $\underset{\sim}{j}$ is recurrent in Y, the amount of time spent in $\underset{\sim}{j}$ is infinite with probability 1, i.e., the set $G_{\underset{\sim}{j}} = \{t;\ Y_t = \underset{\sim}{j}\}$ has infinite length a.s. Therefore the number of T_n^e that fall in $G_{\underset{\sim}{j}}$ must also be infinite. Thus, the number of times $X_n = \underset{\sim}{j}$ is infinite a.s. So $\hat{X}$ is recurrent. A similar argument shows $\hat{X}$ is irreducible.

For any $\underset{\sim}{i},\underset{\sim}{j} \in E$,

$$\begin{aligned} P\{X_1 = \underset{\sim}{j} | X_o = \underset{\sim}{i}\} &= P\{Y(T_1^{e}-) = \underset{\sim}{j} | Y_o = \underset{\sim}{i}\} \\ &= \int_0^{\infty} P\{Y_{t^-} = \underset{\sim}{j} | T_1^e = t, Y_o = \underset{\sim}{i}\} dP\{T_1^e = t | Y_o = \underset{\sim}{i}\} \\ &= \int_0^{\infty} P\{Y_{t^-} = \underset{\sim}{j} | Y_o = \underset{\sim}{i}\} dP\{T_1^e = t\} \end{aligned}$$

since T_1 and Y are independent

$$= \int_0^\infty P_t(\underset{\sim}{i},\underset{\sim}{j})\lambda e^{-\lambda t}dt$$

$$= \lambda U^\lambda(\underset{\sim}{i},\underset{\sim}{j}).$$

Thus, $\hat{X}$ has transition matrix defined by $\lambda U^\lambda(\underset{\sim}{i},\underset{\sim}{j})$, where U^λ is the λ potential of the process Y. It is known (cf. Çinlar (1975) pg. 265) that

$$\pi A = 0 \text{ iff } \pi\lambda U^\lambda = \pi$$

where A is given by equation (2.2). Thus, Y and $\hat{X}$ have the same invariant distribution. □

3.4. The Z Process.

It is shown below that the Z process, which is the queue length process embedded at arrivals to Q_1, is a Markov chain, theorem (4.1). It is also shown that the Z process and the $\hat{X}$ process have the same limiting distribution, theorem (4.3).

Let $N^a = \{N_t^a; t \geq 0\}$ be the Poisson arrival process to Q_1, and let $T^a = \{T_n^a; n \in \mathbb{N}\}$ be the sequence of arrival times. It is apparent that, for each $n \in \mathbb{N}$, T_n^a is a stopping time for Y. Let $Z = \{Z_n; n \in \mathbb{N}\}$ $= \{Y(T_n^a-); n \in \mathbb{N}\}$. Thus, Z is the state of Y the instant before the nth arrival to Q_1. It follows that Z is a Markov chain.

Theorem 4.1. $Z = \{Z_n; n \in \mathbb{N}\}$ is an irreducible, homogeneous Markov chain.

Proof. $P\{Z_{n+1} = \underset{\sim}{j} | Z_o, \cdots, Z_n = \underset{\sim}{i}\} = P\{Y(T_{n+1}^a-) = \underset{\sim}{j} | Y(T_o^a-), \cdots, Y(T_n^a-) = \underset{\sim}{i}\}$

$$= P\{Y(T_1^a-) = \underset{\sim}{j} | Y(T_o^a-) = \underset{\sim}{i}\}$$

by the strong Markov property and the homogeneity of Y

$$= P\{Z_1 = \underset{\sim}{j} | Z_o = \underset{\sim}{i}\}.$$

Thus, Z is a homogeneous Markov chain.

For any state $\underset{\sim}{j} \in E$, the expected number of returns to $\underset{\sim}{j}$ starting from $\underset{\sim}{j}$ is,

$$E\{\sum_{n=1}^{\infty} 1_{\{\underset{\sim}{j}\}}(Y(T_n^a)) \cdot 1_{T^a}(T_n) | Y_o = \underset{\sim}{j}\} = \sum_{n=1}^{\infty} P\{Y(T_n^a) = \underset{\sim}{j},\ T_n \in T^a | Y_o = \underset{\sim}{j}\}$$

by the monotone convergence theorem;

$$= \sum_{n=1}^{\infty} \overset{n}{Q}(\underset{\sim}{j},\underset{\sim}{j}) \frac{\lambda}{\gamma(\underset{\sim}{j})}$$

$$= \infty$$

since, for all $\underset{\sim}{j} \in E$, $\frac{\lambda}{\gamma(\underset{\sim}{j})} > 0$ and $\underset{\sim}{j}$ recurrent in Y implies $\sum_{n=1}^{\infty} \overset{n}{Q}(\underset{\sim}{j},\underset{\sim}{j}) = \infty$. Thus, all states $\underset{\sim}{j}$ are recurrent. Y irreducible implies that, for all $\underset{\sim}{i},\underset{\sim}{j} \in E$, there exists an $n \in \mathbb{N}$ such that $\overset{n}{Q}(\underset{\sim}{i},\underset{\sim}{j}) > 0$. Thus, there is an $n \in \mathbb{N}$ such that $\overset{n}{Q}(\underset{\sim}{i},\underset{\sim}{j})\frac{\lambda}{\gamma(\underset{\sim}{j})} > 0$. Therefore Z is irreducible. □

<u>Theorem</u> 4.2. The processes N^a and N^e are independent.

<u>Proof</u>. For any $i,j \in \mathbb{N}$,

$$P\{N_t^a = i, N_t^e = j\} = E\{1_{\{i\}}(N_t^a) \cdot 1_{\{j\}}(N_t^e)\}$$

$$= E\{E\{1_{\{i\}}(N_t^a) \cdot 1_{\{j\}}(N_t^e) | Y_u;\ u \le t\}\}$$

$$= E\{1_{\{i\}}(N_t^a) E\{1_{\{j\}}(N_t^e) | Y_u;\ u \le t\}\}$$

since N_t^a is completely determined by $\{Y_u;\ u \le t\}$

$$= E\{1_{\{i\}}(N_t^a)\}E\{1_{\{j\}}(N_t^e)\}$$

since N_t^e is independent of $\{Y_u;\ u \le t\}$

$$= P\{N_t^a = i\}P\{N_t^e = j\}. \quad \square$$

<u>Theorem</u> 4.3. $P_\pi\{Z_n = \underset{\sim}{j}\} = P_\pi\{\hat{X}_n = \underset{\sim}{j}\}$ where $P_\pi\{\cdot\} = \sum_{i\in E} P\{\cdot\,|Y_o = \underset{\sim}{i}\}\pi(\underset{\sim}{i})$.

<u>Proof</u>. Let $T^a = \{T_n^a\}$, $T^e = \{T_n^e\}$, and $\hat{T} = T^a \cup T^e = \{\hat{T}_n\}$ where $T_1 = \min\{T_1^a, T_1^e\}$ and $\hat{T}_1 \le \hat{T}_2 \le \cdots$. Since N^a and N^e are independent Poisson processes each with mean λ^{-1}, it follows from Çinlar (1975, pg. 87) that N, the superposition of N^a and N^e, is a Poisson process with mean $(2\lambda)^{-1}$. Let $\hat{Y}_n = Y(\hat{T}_n-)$. Then

$$P_\pi\{\hat{Y}_n = \underset{\sim}{j}\} = P_\pi\{\hat{Y}_n = \underset{\sim}{j}\}P_\pi\{1_{T^a}(\hat{T}_n)|Y_n = \underset{\sim}{j}\} + P_\pi\{\hat{Y}_n = \underset{\sim}{j}\}P_\pi\{1_{T^e}(\hat{T}_n)|\hat{Y}_n = \underset{\sim}{j}\}$$

since by construction of N, the nth jump of $\hat{Y}_n$ cannot be at both N^e and N^a. By construction of N,

$$P_\pi\{1_{T^a}(\hat{T}_n)|\hat{Y}_n = \underset{\sim}{j}\} = P_\pi\{1_{T^e}(\hat{T}_n)|\hat{Y}_n = \underset{\sim}{j}\} = 1/2.$$

Thus, for all n,

$$P_\pi\{\hat{Y}_n = \underset{\sim}{j},\ 1_{T^a}(\hat{T}_n)\} = P_\pi\{\hat{Y}_n = \underset{\sim}{j},\ 1_{T^e}(\hat{T}_n)\}.$$

Since,

$$P_\pi\{1_{T^a}(\hat{T}_n)\} = P_\pi\{1_{T^e}(\hat{T}_n)\} = 1/2,$$

if follows that,

$$P_\pi\{\hat{Y}_n = \underset{\sim}{j} \,|\, 1_{T^a}(\hat{T}_n)\} = P_\pi\{\hat{Y}_n = \underset{\sim}{j} \,|\, 1_{T^e}(\hat{T}_n)\},$$

or

$$P_\pi\{Z_n = \underset{\sim}{j}\} = P_\pi\{\hat{X}_n = \underset{\sim}{j}\}. \quad \square$$

As a consequence of this result it follows that Z and $\hat{X}$ have the same invariant distribution.

3.5. <u>A Nonequilibrium Distribution</u>.

It is shown below that if the Y process is in equilibrium at some time t_o and if T_n^a is the first arrival epoch after t_o, then $P\{Y(T_n^a-) = \underset{\sim}{i}\}$ is not equal to the equilibrium probability. This result has implications to the way the system is simulated as discussed in the summary.

Suppose at t_o, Y is in equilibrium. Let T be the time of the next arrival to Q_1. Define Y_t^1 to be the queue length in Q_1 at time t. In order for Y to be in equilibrium the instant before the next arrival to Q_1, for every $i \in \mathbb{N}$, $P\{Y_{t_o}^1 \leq i\} = P\{Y_{T^-}^1 \leq i\}$. For every sample path ω, we have that $Y_{t_o}^1(\omega) \leq Y_{T^-}^1(\omega)$. Thus, for each $i \in \mathbb{N}$,

$$\{\omega;\ Y_{t_o}^1(\omega) \leq i\} \quad \{\omega;\ Y_{T^-}^1(\omega) \leq i\}.$$

Therefore,

$$P\{\omega;\ Y_{t_o}^1(\omega) \leq i\} \leq P\{\omega;\ Y_{T^-}^1(\omega) \leq i\}. \tag{5.1}$$

In order for equation (5.1) to be an equality,

$$P(\{\omega;\ Y_{T^-}^1(\omega) \leq i\} - \{\omega;\ Y_{t_o}^1(\omega) \leq i\}) = 0.$$

However, this implies that

$$0 = P(\{\omega;\ Y_{T^-}^{1}(\omega) \le i\} \cap \{\omega;\ Y_{t_o}^{1}(\omega) > i\})$$

$$\ge P(\{\omega;\ Y_{T^-}^{1}(\omega) \le i\} \cap \{\omega;\ Y_{t_o}^{1}(\omega) = i+1\})$$

$$= P(\{\omega;\ Y_{T^-}^{1}(\omega) < Y_{t_o}^{1}(\omega)\} \cap \{\omega;\ Y_{t_o}^{1}(\omega) = i+1\})$$

$$= (1-e^{-\mu_1 t})\rho^{i+1}(1-\rho) \quad \text{where } \rho = \lambda/\mu_1$$

$$> 0 \quad \text{for all } t > 0,\ i \in \mathbb{N}, \text{ a contradiction.}$$

Thus,

$$P\{\omega;\ Y_{t_o}^{1}(\omega) \le i\} < P\{\omega;\ Y_{T^-}^{1}(\omega) \le i\}.$$

Therefore, if at t_o, Y is in equilibrium, then the distribution of Y at T_n^{a-} is not the equilibrium distribution. This result makes intuitive sense, since between t_o and T_n^{a-} departures from Q_1 can occur.

3.6. Summary.

In section (3.1) we showed that the queue length process, Y, of the three queue network is an irreducible, recurrent, Markov process. Thus, simulating the three queue network in equilibrium is done by simulating the Markov process, Y, in equilibrium. The distribution of time spent in a state is given by equation (1.4). In section (3.2) the transition probabilities of the underlying Markov chain, X, of Y are developed, equation (2.1). Equations (1.4) and (2.1) are important in constructing the next event generator of the simulation, as will be shown in section (4.4).

In section (3.4) it was shown that the queue length process embedded

at arrival epochs, Z, is a Markov chain with the same equilibrium distribution as Y. In section (3.5) we showed that if Y is in equilibrium at an arbitary time t_o, then the queue length distribution the instant before the next arrival to Q_1 is not the equilibrium distribution. The results of section (3.4) and (3.5) are important in initializing the simulation, as will be shown in section (4.2).

CHAPTER 4

THE SIMULATION APPROACH

4.1. Introduction.

In this section we will construct an efficient simulation of the three queue network from which the correlation between S_1 and S_3 can be correctly analyzed. In constructing such a simulation, we take advantage of the properties of the Markov process, Y, developed in Chapter 3. The method of initializing the simulation, discussed in section (4.2) is correct, as well as, efficient. We force a tagged customer to take the route Q_1,Q_2,Q_3 and calculate his sojourn times at these queues. In section (4.3) why the tagged customer is forced to take the route Q_1,Q_2,Q_3 is discussed. After the tagged customer leaves the network, we stop the simulation, reinitialize it, and run it again. Also in section (4.3) we explain why the simulation is reinitialized each time the tagged customer leaves the network. Finally, in section (4.4) the simulator's next event generator is discussed. In that section we use the results of section (3.2).

4.2. Initialization of the Simulation.

For each iteration of the simulation the queue length at each queue is initialized according to the network's equilibrium queue length distribution, equation (2.2.3). Thus, the initial distribution of customers at each queue is geometric. For each i, $i=1,2,3$, to generate the initial queue length at Q_i it suffices to use a geometric random number generator with parameter ρ_i, where $\rho_1 = \lambda/\mu_1$, $\rho_2 = p\lambda/\mu_2$, and $\rho_3 = \lambda/\mu_3$. In this simulation the geometric random number generator

used is the IMSL routine GGEM.

One has two choices as to when to add the tagged customer to Q_1. One can either add the tagged customer to Q_1 and then start the simulation or one can start the simulation, wait until an arrival to Q_1 is generated, and let this arrival be the tagged customer. In section (3.2) it was shown that the queue length process embedded just before arrivals to Q_1, Z, is a Markov chain with the same equilibrium distribution as Y. Thus, it is sufficient to add the tagged customer to Q_1 and then start the simulation. In section (3.5) it was shown that if at some time, t_o, the network is in equilibrium, then the next arrival to Q_1 does not see an equilibrium queue length distribution at Q_1. Thus, if one starts the simulation and calls the next arrival to Q_1 the tagged customer, then one will not be simulating a network in equilibrium. Thus, it is necessary and sufficient to add the tagged customer to Q_1 and then start the simulation.

4.3. Independence of Output and Customer Routing.

In section (4.1) it was stated that the simulation was reinitialized each time a tagged customer left the network. So, for each tagged customer, c_i, $i = 1,\cdots,M$, we are simulating a segment of a distinct sample path, ω_i, of Y. Thus, if we take observations $\underset{\sim}{x}_i$ on ω_i, $i = 1,\cdots,M$, the $\underset{\sim}{x}_i$ are mutually independent. Further, since each c_i sees Y in equilibrium at their arrival to Q_1, the observations $\underset{\sim}{x}_i$ are identically distributed. The fact that the $\underset{\sim}{x}_i$ are observations on a sequence of independent identically distributed random variables is required to compute and to analyze the statistics of the simulation,

Chapter 5.

The purpose of the simulation is to determine the correlation between S_1 and S_3 given the tagged customer takes the route $\ell_1 = Q_1Q_2Q_3$. It can be shown from Simon and Foley (1979) that S_1 and S_3 are independent given the customer takes the route $\ell_2 = Q_1Q_3$. Thus, we are interested in only those tagged customers which take the route ℓ_1. By forcing the tagged customers to take the route ℓ_1, the number of iterations the simulation requires over the number of iterations the simulation would require if we let the tagged customer choose which path to take, is reduced. For example, for a sample of 1000 tagged customers who take the route ℓ_1, in a network with switching parameter $p = .01$, by forcing the tagged customers to take route ℓ_1 the simulation requires only 1000 iterations. However, by not forcing the tagged customers to take the route ℓ_1 the expected number of iterations the simulation requires is 100,000.

4.4. Generation of the Next Event.

Let T_n, $n \in \mathbb{N}$, be the time of the nth jump of the process Y and X_n; $n \in \mathbb{N}$, be the state Y enters at T_n. Then from Çinlar (1975, pg. 247).

$$P\{X_{n+1} = \underset{\sim}{j},\ T_{n+1} - T_n > t \mid X_o, \cdots, X_n = \underset{\sim}{i},\ T_o, \cdots, T_n\} = Q(\underset{\sim}{i},\underset{\sim}{j})e^{-\gamma(\underset{\sim}{i})t} \tag{4.1}$$

where $Q(\underset{\sim}{i},\underset{\sim}{j})$ is given by equation (3.2.1) and $e^{-\gamma(\underset{\sim}{i})t}$ by equation (3.1.4).

It follows that the process $X = \{X_n;\ n \in \mathbb{N}\}$ is a Markov chain with transition probabilities given by the $Q(\underset{\sim}{i},\underset{\sim}{j})$, see section (3.2). So the next event is determined by the Markov chain X. Thus, to generate the

type of the next event it suffices to generate a uniform random number, r, such that:

(i) if $0 \leq r < \frac{\lambda}{\gamma(\underset{\sim}{i})}$ the next event is an arrival to Q_1;

(ii) if $\frac{\lambda}{\gamma(\underset{\sim}{i})} \leq r < \frac{(\lambda+p\mu_1(i_1))}{\gamma(\underset{\sim}{i})}$ the next event is a departure from Q_1 who goes to Q_2;

(iii) if $\frac{(\lambda+p\mu_1(i_1))}{\gamma(\underset{\sim}{i})} \leq r < \frac{(\lambda+\mu_1(i_1))}{\gamma(\underset{\sim}{i})}$ the next event is a departure from Q_1 who goes to Q_3;

(iv) if $\frac{\lambda+\mu_1(i_1)}{\gamma(\underset{\sim}{i})} \leq r < \frac{\lambda+\mu_1(i_1)+\mu_2(i_2)}{\gamma(\underset{\sim}{i})}$ the next event is a departure from Q_2 who goes to Q_3;

(v) if $\frac{\lambda+\mu_1(i_1)+\mu_2(i_2)}{\gamma(i)} \leq r \leq 1$ the next event is a departure from Q_3 who leaves the network.

The uniform random number generator used is the ISML routine GGUM.

Further, from equation (3.1), it follows that

$$P\{T_{n+1} - T_n > t | X_o, \cdots, X_n = \underset{\sim}{i},\ X_{n+1} = \underset{\sim}{j},\ T_o, \cdots, T_n\} = e^{-\gamma(\underset{\sim}{i})t}.$$

Thus, if at the last event Y is in state $\underset{\sim}{i}$ the time until the next event is determined by an exponential distribution with mean $\gamma(\underset{\sim}{i})^{-1}$. Thus, to generate the time of the next event it suffices to use an exponential random generator with mean $\gamma(\underset{\sim}{i})^{-1}$. The exponential random number generator used is the IMSL routine GGEXP.

4.5. Summary.

In this chapter, important concepts of the construction of the simulation were discussed. First due to the construction of the simulation the data are independent and identically distributed. This is necessary for the analysis done in the next chapter. Since the equilibrium distribution of Y is known, the initial distribution of customers can be calculated directly instead of by simulating the network for a period of time to approximate an initial distribution. Since we are trying to gather data concerning a customer's total sojourn time distribution in a network in equilibrium it is essential that the network be in equilibrium when the tagged customer arrives. In section (4.2) it was explained when the queue length distribution was and was not in equilibrium with respect to the arrival of the tagged customer. Finally, since the queue length process is Markov, instead of creating a large next event file one needs the transition probabilities and the exponential distribution of time the process spends in state of E. Flow charts, program listings, and program discriptions are listed in appendix 1.

CHAPTER 5

ANALYSIS OF THE SIMULATION

5.1. Introduction.

For a given set of parameter settings we simulate the three queue network of section (1.3). The simulation is run for 1000 iterations. For each iteration, i, the network is initialized according to it's equilibrium queue length distribution. Next, a tagged customer is added to Q_1. The simulation is run with all customers, except the tagged customer, moving through the network according to it's parameter settings. The tagged customer always takes the route Q_1, Q_2, Q_3. The sojourn times, S_1^i, S_2^i, S_3^i, of the tagged customer in each queue, and his total sojourn time TS_1^i are recorded. Once the tagged customer leaves the network, an iteration of the simulation is complete. Details of this procedure are found in Chapter 4.

For $\lambda = 4.0$, $\mu_i = 5.0$, $i = 1,2,3$, the simulation is run for $p = 0.00$, 0.01, 0.05, 0.10, 0.25, 0.50, and 1.00. For $p = 1.00$, we have a network of three tandem queues. For this case, we should have that S_1, S_2, and S_3 are mutually independent random variables, Reich (1963). For $p = 0.01$, 0.05, 0.10, 0.25, and 0.50, according to Simon and Foley (1979), section (2.5), S_1 and S_3 are dependent. One purpose of this simulation study is to determine how S_1 and S_3 are correlated for these values of p. For $p = 0.00$ we are not simulating two tandem queues since the tagged customer still takes the route Q_1, Q_2, Q_3. For this value of p we are trying to determine the correlation between S_1 and S_3 as p decreases to zero.

The analysis of the simulation consists of two parts. First, a test is done to determine whether the correlation between S_1 and S_3 is significant. Second, a test is run to determine whether the sample distribution of the total sojourn time is different from one which supposes that S_1 and S_3 are independent.

5.2. Analysis of Correlation.

The statistic used to determine the correlation of a customer's sojourn times in Q_1 and Q_3 is the correlation coefficient. The correlation coefficient is a descriptive index on two sets of data, (X,Y), the value of which serves to specify the dependence exhibited by the data between the variables X and Y. Mathematically the correlation coefficient, r, between two sets of data (X_i,Y_i) $i = 1,\cdots,N$ is given by

$$r = \sqrt{\left(\sum_{i=1}^{N} X_iY_i - \left(\sum_{i=1}^{N} X_i\right)\left(\sum_{i=1}^{N} Y_i\right)/N\right)} \Big/ \sqrt{\left(\sum_{i=1}^{N} X_i^2 - \left(\sum_{i=1}^{N} X_i\right)^2/N\right)\left(\sum_{i=1}^{N} Y_i^2 - \left(\sum_{i=1}^{N} Y_i\right)^2/N\right)}.$$

The coefficient, r, must have the property that $-1 \leq r \leq 1$. A positive value of r implies positive correlation, that is, a large value in X implies a large value in Y. A negative value for r implies negative correlation, that is, a large value in X implies a small value in Y. A value of $r = 0$ implies that there is no correlation. This does not imply, however, that X and Y are independent sets of data. However, a value of r that is not zero implies that the data sets X and Y are correlated and thus cannot be independent.

Let,

X_i be the sojourn time of the ith customer in Q_1;

Y_i be the sojourn time of the ith customer in Q_2;

Z_i be the sojourn time of the ith customer in Q_3;

r_1 be the correlation coefficient between the data sets (X_i, Y_i);

r_2 be the correlation coefficient between the data sets (X_i, Z_i);

r_3 be the correlation coefficient between the data sets (Y_i, Z_i).

In this section we will show that r_1 and r_2 are not statistically greater than zero.

To test the hypothesis $r_i = 0$, $i = 1,2,3$, Fisher's Z transformation is used to change the correlation coefficient, r_i, into a normal Z statistic. This relationship is given by

$$Z = 1/2 \ln\left(\frac{1+r_i}{1-r_i}\right).$$

The standard deviation of the Z statistic is given by

$$\delta_Z = 1/\sqrt{n-3}.$$

Thus, our test of hypothesis becomes:

(i) H_o: $r_i = 0 \quad i = 1,2,3$ (null hypothesis);

(ii) H_1: $r_i \neq 0 \quad i = 1,2,3$ (alternative hypothesis);

(iii) Critical region, $\alpha = .05$;

(iv) Test statistic

$$Z = Z/\delta_Z = 1/2 \ln\left(\frac{1+r_i}{1-r_i}\right)\sqrt{n-3} \;;$$

(v) Conclusion: Accept H_o if $Z \leq 1.96$ otherwise accept H_1.

A $1-\alpha$ confidence interval for the Z statistic is given by the formula

$$(Z - \frac{Z_{\alpha/2}}{\sqrt{n-3}} , Z + \frac{Z_{\alpha/2}}{\sqrt{n-3}}).$$

To get the corresponding interval for the correlation coefficient insert the left and right hand limits of the above interval into

$$r = \frac{e^{2Z}-1}{e^{2Z}+1} .$$

For $\alpha = .05$ if 0 does not lie in the interval one can be 95% certain that r_i is nonzero. If the interval is positive then one can be 95% certain that r_i is positive.

Since each run consists of 1000 observations, it follows that if $|r_i| > .062$, $i = 1,2,3$, then the null hypothesis is rejected. To see this, note that the null hypothesis is rejected if

$$Z < 1/2 \ln(\frac{1+r_i}{1-r_i})\sqrt{n-3} \quad \text{or} \quad Z > -1/2 \ln(\frac{1+r_i}{1-r_i})\sqrt{n-3} .$$

Thus, we reject the null hypothesis if

$$\ln(\frac{1+r_i}{1-r_i}) > 2Z/\sqrt{n-3} .$$

So, we reject the null hypothesis if

$$\frac{1+r_i}{1-r_i} > e^{2Z/\sqrt{n-3}}$$

or

$$r_i > \frac{e^{2Z/\sqrt{n-3}} - 1}{e^{2Z/\sqrt{n-3}} + 1} .$$

Inserting in $Z = 1.96$ and $n = 1000$ we obtain the desired result. The

$1 - \alpha$ confidence intervals for $\alpha = .05$ are given by $Z \pm .062$ where

$$Z = 1/2 \ln(\frac{1+r}{1-r})\sqrt{n-3} .$$

Putting the limit points of the above interval into

$$r = e^{2Z}-1/e^{2Z}+1$$

yields the $1 - \alpha$ confidence interval for r. The results of the tests are listed in Table 2.1, Table 2.2, and Table 2.3.

From Table 2.1 the only significant correlations between sojourn times in Q_1 and Q_2 oocur when p (the switching probability) takes the values .01 and .05. In each case significant correlation occured for just one of the five runs. From Table 2.2 there were no significant correlations between sojourn times in Q_2 and Q_3. However, for all values of p except .05 and 1.0 there were runs with significant correlation between sojourn times in Q_1 and Q_3. Since, for each case in which there were runs of significant correlation there were also runs in which the correlation was not significant further testing needed to be done.

To further test for significant correlation between the sojourn times in Q_1 and Q_3 the same z-transformation test described above was used with the following modification. Instead of testing the correlation coefficient for each run separately the average over the five runs was taken. For example, from Table 2.3 for $p = 0.0$ the new r value would be

$$r = \{.0582 + .0532 + (-.0209) + .0708\}/5 = .03226.$$

Note that the sample size of this test is $n = 5000$ rather than $n = 1000$.

TABLE 2.1

ANALYSIS OF CORRELATION BETWEEN QUEUE 1 AND QUEUE 2

p	r	sig.	con. int.
0.00	-.0591	no	(-.0121,.0029)
0.00	-.0237	no	(-.0857,.0383)
0.00	.0030	no	(-.0590,.0650)
0.00	-.0304	no	(-.0924,.0316)
0.00	-.0187	no	(-.0807,.0433)
0.01	-.0019	no	(-.0639,.0601)
0.01	-.0759	yes	(-.1378,-.0139)
0.01	-.0253	no	(-.0873,.0367)
0.01	.0027	no	(-.0593,.0647)
0.01	-.0435	no	(-.1055,.0185)
0.05	-.0189	no	(-.0809,.0431)
0.05	-.0177	no	(-.0797,.0443)
0.05	.0123	no	(-.0497,.0743)
0.05	-.0240	no	(-.0860,.0380)
0.05	.0165	no	(-.0455,.0785)
0.10	-.0225	no	(-.0845,.0395)
0.10	.0061	no	(-.0559,.0681)
0.10	.0295	no	(-.0385,.0915)
0.10	-.0075	no	(-.0695,.0545)
0.10	.0357	no	(-.0263,.0977)
0.25	-.0431	no	(-.1051,.0189)
0.25	.0288	no	(-.0332,.0908)
0.25	-.0169	no	(-.0789,.0451)
0.25	.0292	no	(-.0328,.0912)
0.25	.0225	no	(-.0395,.0845)
0.50	-.0499	no	(-.1119,.0121)
0.50	-.0095	no	(-.0715,.0525)
0.50	.0728	yes	(.0108,.1348)
0.50	-.0329	no	(-.0949,.0291)
0.50	.0150	no	(-.0470,.0770)
1.00	-.0604	no	(-.1224,.0016)
1.00	.0206	no	(-.0414,.0826)
1.00	-.0184	no	(-.0804,.0436)
1.00	.0512	no	(-.0108,.0512)
1.00	-.0432	no	(-.1052,.0188)

TABLE 2.2

ANALYSIS OF CORRELATION BETWEEN QUEUE 2 AND QUEUE 3

p	r	sig.	con. int.
0.00	-.0260	no	(-.088,.036)
0.00	-.0261	no	(-.0881,.0359)
0.00	.0030	no	(-.0650,.0590)
0.00	-.0193	no	(-.0813,.0427)
0.00	-.0461	no	(-.1081,.0159)
0.01	.0061	no	(-.0559,.0681)
0.01	-.0431	no	(-.1051,.0189)
0.01	-.0200	no	(-.0820,.0420)
0.01	-.0294	no	(-.0914,.0326)
0.01	-.0267	no	(-.0887,.0353)
0.05	-.0027	no	(-.0647,.0593)
0.05	.0023	no	(-.0597,.0643)
0.05	-.0068	no	(-.0688,.0552)
0.05	.0269	no	(-.0351,.0889)
0.05	.0459	no	(-.0161,.1079)
0.10	.0062	no	(-.0558,.0682)
0.10	-.0343	no	(-.0963,.0277)
0.10	.0067	no	(-.0553,.0687)
0.10	-.0369	no	(-.0989,.0251)
0.10	-.0285	no	(-.0335,.0905)
0.25	.0237	no	(-.0383,.0857)
0.25	-.0353	no	(-.0973,.0267)
0.25	.0394	no	(-.0226,.1014)
0.25	.0054	no	(-.0566,.0674)
0.25	-.0205	no	(-.0825,.0485)
0.50	.0198	no	(-.0422,.0818)
0.50	.0174	no	(-.0446,.0794)
0.50	-.0216	no	(-.0836,.0404)
0.50	-.0217	no	(-.0837,.0403)
0.50	-.0294	no	(-.0326,.0914)
1.00	.0079	no	(-.0541,.0699)
1.00	.0080	no	(-.0540,.0700)
1.00	-.0144	no	(-.0764,.0476)
1.00	-.0126	no	(-.0746,.0504)
1.00	-.0434	no	(-.1054,.0186)

TABLE 2.3

ANALYSIS OF CORRELATION BETWEEN QUEUE 1 AND QUEUE 3

p	r	sig.	con. int.
0.00	.0582	no	(-.0038,.1202)
0.00	.0532	no	(-.0088,.1152)
0.00	-.0209	no	(-.0829,.0411)
0.00	.0708	yes	(.0088,.1328)
0.00	.0347	no	(-.0273,.0967)
0.01	.0735	yes	(.0115,.1355)
0.01	.0528	no	(-.0092,.1768)
0.01	-.0080	no	(-.0700,.0540)
0.01	.0546	no	(-.0074,.1166)
0.01	-.0246	no	(-08660,.03774)
0.05	.0599	no	(-.0021,.1219)
0.05	.0651	yes	(.0031,.1271)
0.05	.0616	no	(-.0004,.1236)
0.05	.0085	no	(-.0535,.0705)
0.05	.1034	yes	(.0414,.1654)
0.10	.0252	no	(-.0368,.0872)
0.10	.0439	no	(-.0181,.1059)
0.10	.0308	no	(-.0312,.0928)
0.10	-.0314	no	(-.0934,.0306)
0.10	.0400	no	(-.0220,.1020)
0.25	-.0310	no	(-.0930,.0310)
0.25	.0023	no	(-.0597,.0643)
0.25	.0520	no	(-.0100,.1140)
0.25	.0758	yes	(.0138,.1378)
0.25	.0915	yes	(.0295,.1535)
0.50	-.0074	no	(-.0546,.0694)
0.50	.0524	no	(-.0096,.1144)
0.50	.0043	no	(-.0577,.0663)
0.50	-.0182	no	(-.0802,.0438)
0.50	.0252	no	(-.0368,.0872)
1.00	.0140	no	(-.0480,.0760)
1.00	.0427	no	(-.0193,.1047)
1.00	-.0440	no	(-.1060,.0180)
1.00	-.0111	no	(-.0731,.0509)
1.00	-.0086	no	(-.0706,.0534)

TABLE 2.4

ANALYSIS OF CORRELATION BETWEEN QUEUE 1 AND QUEUE 3
WITH RESPECT TO THE AVERAGES OF THE RUNS

p	r	sig.	con. int.
0.00	.03226	yes	(.0049,.0603)
0.01	.02966	yes	(.00196,.05736)
0.05	.05970	yes	(.0320,.08740)
0.10	.02170	no	(-.0060,.04940)
0.25	.03812	yes	(.01042,.06582)
0.50	.00622	no	(-.02148,.03392)
1.00	-.0014	no	(.0291,.0263)

It follows from the formula that

$$r = e^{aZ}-1/e^{aZ}+1 \quad a = 2/\sqrt{n-3}, \quad Z = 1.96$$

that if $r > .0277$ then the null hypothesis $r_3 = 0$ is rejected. The results of this test are listed in Table 2.4.

In the cases of $p = 0.00$, 0.01, 0.05, and 0.25 the tests show that the correlation is significant. Further, the confidence intervals show that correlation is positive in these cases.

5.3. Testing the Total Sojourn Time Distribution.

Let JN specify the three queue network, see section (3.1). Suppose that for a customer, c, c's sojourn time in Q_1 and Q_3 are independent given he takes the route Q_1,Q_2,Q_3. What would c's total sojourn time distribution be? Let

$$\phi_1(t) = P\{S_1 \le t\} = 1 - e^{-at} \quad \text{where } a = \mu_1 - \lambda;$$

$$\phi_2(t) = P\{S_2 \le t\} = 1 - e^{-bt} \quad \text{where } b = \mu_2 - p\lambda;$$

$$\phi_3(t) = P\{S_3 \le t\} = 1 - e^{-ct} \quad \text{where } c = \mu_3 - \lambda.$$

Proposition 3.1. If the total sojourn times at the three queues were mutually independent, which they are not, then the total sojourn time distribution of a customer taking the route Q_1,Q_2,Q_3 where $a = c$, a and c defined above, would be

$$P\{S_1+S_2+S_3 \le t\} = 1 - \frac{a^2}{(b-a)^2}\, e^{-bt} - \left[\frac{b^2+ba((b-a)t-2)}{(b-a)^2}\right] e^{-at}\,.$$

<u>Proof</u>. From equation (2.3.3)

$$P\{S_1 + S_2 + S_3 \leq t | \ell_1\} = (\phi_1 * \phi_2 * \phi_3)(t)$$

$$= \int_{[0,t]} \int_{[0,t-x]} \phi_3(t-x-y)\phi_2(dy)\phi_1(dx).$$

We first evaluate

$$\int_{[0,t-x]} \phi_3(t-x-y)\phi_2(dy).$$

Let $u = t - x$, then

$$\int_{[0,t-x]} \phi_3(t-x-y)\phi_2(dy) = \int_0^u \phi_3(u-y)\phi_2(dy)$$

$$= \int_0^u (1 - e^{-a(u-y)})be^{-by}dy$$

$$= 1 - e^{-bu} + \frac{b}{(b-a)} e^{-au}[e^{-(b-a)u}]$$

$$= 1 + \frac{a}{(b-a)} e^{-bu} - \frac{b}{(b-a)} e^{-au}.$$

Thus, the total sojourn time distribution is

$$\int_0^t (1 + \frac{a}{(b-a)} e^{-b(t-x)} - \frac{b}{(b-a)} e^{-a(t-x)})ae^{-at}dt$$

$$= 1 - e^{-at} + \frac{a^2e^{-at}}{(b-a)^2} - \frac{a^2}{(b-a)^2} e^{-bt} - \frac{bat}{(b-a)} e^{-at}$$

$$= 1 - \frac{a^2}{(b-a)^2} e^{-bt} - [\frac{b^2+ba((b-a)t-2)}{(b-a)^2}]e^{-at}.$$

Figure (3.1) gives one graph of formula (3.1). We have superimposed the corresponding distribution function of $S_1 + S_2 + S_3$ on that graph. Differences of the two plots appear to be minor, but one notes that the tails are somewhat different. To further illustrate this discrepancy, the difference between the two distributions is shown in figure (3.2). It would appear from this, that the two distributions are not the same but the differences are small. A Kolmogorov-Smirnov goodness of fit test was run to try to substantiate the visual impression that the sample total sojourn time of the simulation differs from the distribution given by Proposition (3.1).

The Kolmogorov-Smirnov test depends on the statistic

$$D_n = \sup_{-\infty<x<\infty} |F_n(x) - F_o(x)|$$

where $F_n(x)$ is the sample distribution and $F_o(x)$ is the distribution of Proposition (2.1). If the null hypothesis is H_o: $F_o = F_n$ and the alternative hypothesis is H_1: $F_o \neq F_n$. With a critical region of α, α = .20, .15, .10, .05, .01, the null hypothesis is accepted if $D_n < D_\alpha$ otherwise it is rejected. For the case of large n, n > 80, D_α is given by

α	.20	.15	.10	.05	.01
D_α	$1.07/\sqrt{n}$	$1.14/\sqrt{n}$	$1.27/\sqrt{n}$	$1.36/\sqrt{n}$	$1.63/\sqrt{n}$.

For the case α = .05 and n = 1000, D_α = .04030. The result of this test are listed in Table 3.1. For the case where α = .05 and p $\neq$ 0 for all runs, buy two, the null hypothesis was accepted. However, for the case where p = 0.0 the null hypothesis was rejected three of the five runs.

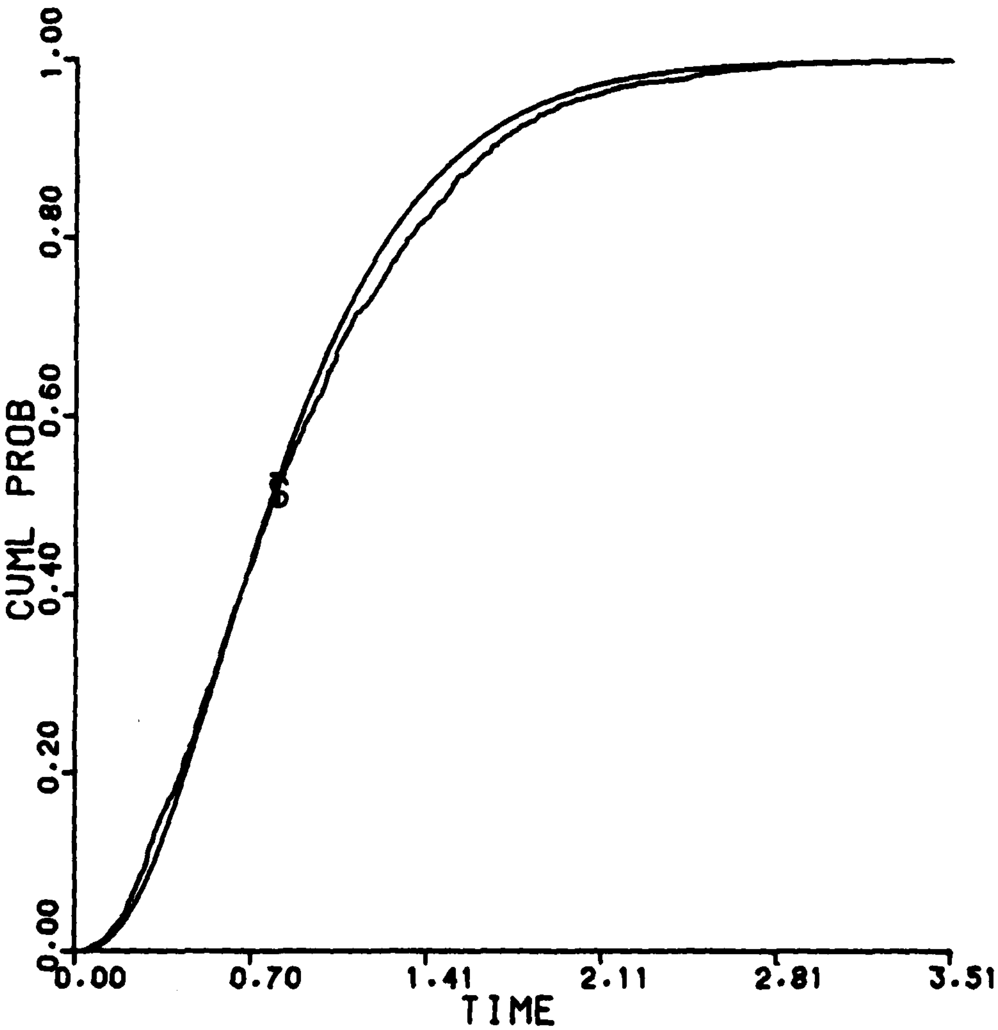


Figure 3.1. A Comparison Between a Sample Distribution, S, and one Assuming Independence, T, for p = .1

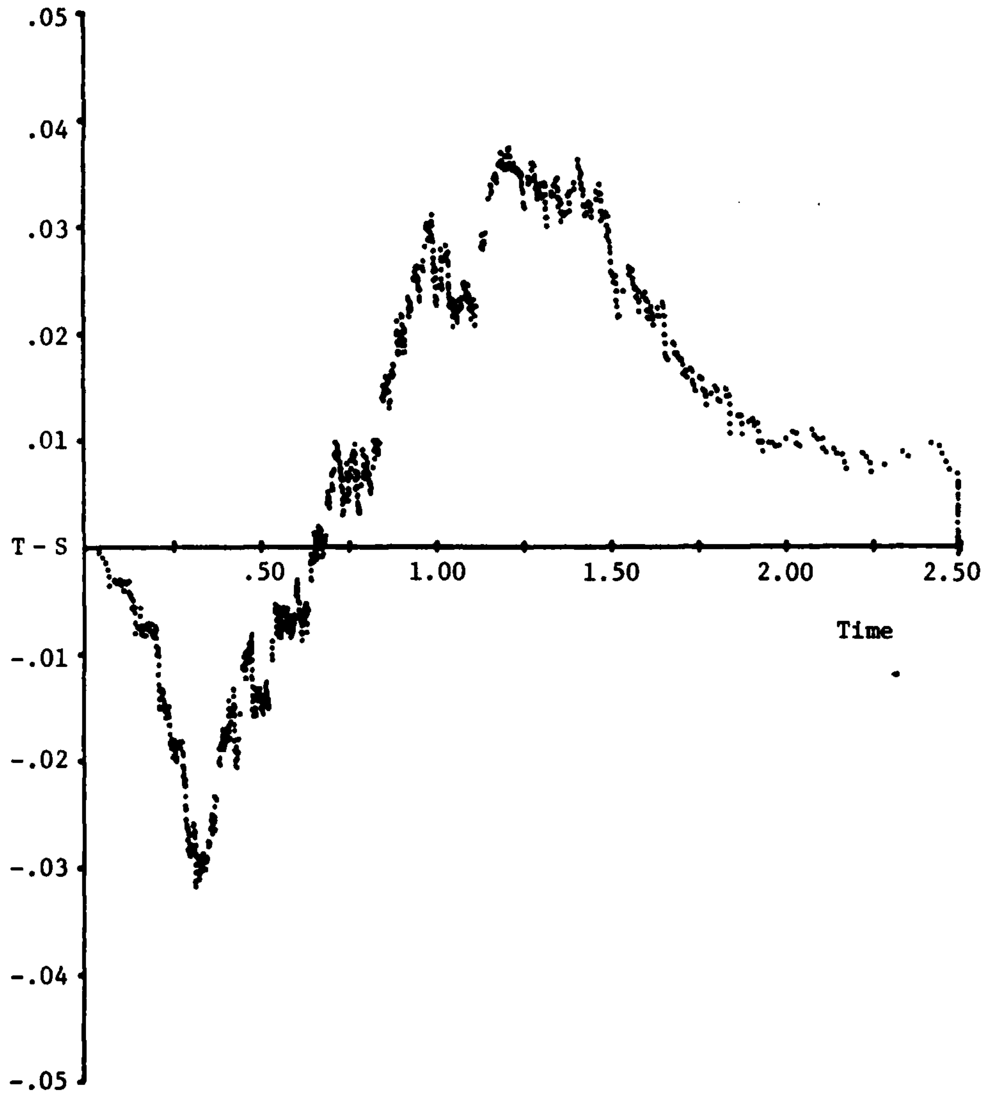


Figure 3.2. The Difference Between T and S

TABLE 3.1

ANALYSIS OF TOTAL SOJOURN TIME DISTRIBUTION

p	D_n	sig.
0.00	.0529	yes
0.00	.0256	no
0.00	.0441	yes
0.00	.0461	yes
0.00	.0255	no
0.01	.0281	no
0.01	.0258	no
0.01	.0392	no
0.01	.0251	no
0.01	.0277	no
0.05	.0163	no
0.05	.0291	no
0.05	.0426	no
0.05	.0352	no
0.05	.0335	no
0.10	.0251	no
0.10	.0243	no
0.10	.0194	no
0.10	.0311	no
0.10	.0307	no
0.25	.0294	no
0.25	.0154	no
0.25	.0393	no
0.25	.0421	no
0.25	.0467	yes
0.50	.0325	no
0.50	.0341	no
0.50	.0513	yes
0.50	.0312	no
0.50	.0141	no
1.00	.0238	no
1.00	.0235	no
1.00	.0200	no
1.00	.0401	no
1.00	.0293	no

Thus, based on this Kolmogorov-Smirnov test, we are unable to verify our visual impression. This is not conclusive evidence that the sample distribution is different from the one assuming independence. However, in view of the results obtained in section (4.1) one is led to believe this is quite likely the case.

CHAPTER 6

SUMMARY AND CONCLUSIONS

6.1. Summary.

Chapter 1 introduced the sojourn time problem. An informal definition of queueing networks was given, as well as, one for the three queue network. Chapter 2 defined Jackson queueing networks and discussed many of their known properties. The most important result concerning this thesis is the Simon and Foley (1979) result in which the authors showed that for the three queue network the sojourn time of a customer at Q_1 and his sojourn time at Q_3 are not independent.

Chapter 3 showed that the queue length process, Y, of the three queue network is a Markov process. The infinitesimal generator, A, of Y was found. The transition matrix of the underlying Markov chain was determined, as well as, the distribution of time spent in a state. Further, it was shown the Y process embedded at arrival instants to Q_1 is a Markov chain with the same equilibrium distribution as Y. Finally, it was shown that if at time t_o Y is in equilibrium then Y will not be in equilibrium at the time of the next arrival to Q_1.

In chapter 4 the simulation was constructed using the properties of the Markov process Y. The simulation was constructed so that the output was a sequence of independent identically distributed observations. The simulation was initialized so that the tagged customer saw a network in equilibrium upon his arrival to Q_1. Finally, the next event generator was constructed using the transition probabilities of the underlying Markov chain of Y, together with the exponential distribu-

tion of time Y spends in a state.

Chapter 5 consisted of the analysis of the simulation. A Fisher Z test was run to determine whether the correlation between S_1 and S_3 was significant. For sample sizes of 1000 we found that the correlation was not significant. A Kolmorgov-Smirnov test found that the sample distribution of the total sojourn time was not significantly different from one assuming that S_1,S_2, and S_3 are mutually independent. Thus, in applications one attempting to model such a network can do so assuming independence.

6.2. Conclusions.

Simon and Foley (1979) show that there is correlation between S_1 and S_2. The tests on the modified Z test in section (5.2) provided evidence that the correlation is small, positive, and a decreasing function of the switching parameter p. Further, graphs of the sample distribution versus the distribution assuming independence show that the sample distribution usually had more mass in its tails providing more evidence that S_1 and S_3 are positively correlated, although the differences were small enough to pass a Kolmorgov-Smirnov test on a sample of size 1000.

6.3. Further Research.

Areas for further research fall into two main categories. First, the simulation needs to be extended to more general acyclic networks. From this extension one can determine whether or not the distribution assuming independence can be substituted for the true, unknown distribution, for modeling purposes. Secondly, analyic work needs to be done to

determine the correlation between sojourn times in Q_1 and Q_3, and more importantly the total sojourn time distribution.

BIBLIOGRAPHY

1. Beutler, F. J. and Melamed, B., (1979), "Decomposition and Customer Streams of Feedback Networks of Queues in Equilibrium", Oper. Res., 26, 1059-1072.

2. Burke, P. J., (1956), "The Output of a Queueing System", Oper. Res., 4, 699-704.

3. Burke, P. J., (1968), "The Output Processes of a Stationary M/M/s Queueing System", Ann. Math. Stat., 39, 1144-1152.

4. Burke, P. J., (1969), "The Dependence of Sojourn Times in Tandem M/M/s Queues", Oper. Res., 17, 754-755.

5. Burke, P. J., (1972), "Output Processes and Tandem Queues", Proceedings of the Symposium on Computer-Communications Networks and Teletraffic, Polytechnic Institute of Brooklyn, April 4-6, 419-428, 1972.

6. Çinlar, E., (1975), Introduction to Stochastic Processes, Prentice Hall, Englewood Cliffs, New Jersey.

7. Disney, R. L., (1975), "Flow in Queueing Networks, A Review and Critique", AIIE Trans., 7, 468-488.

8. Foley, R. C., (1979), "The M/G/1 Queue with Delayed Feedback", Tech report VTR 8010, Dept. of I.E.O.R., V.P.I. & S.U., Blacksburg, Virginia.

9. Jackson, J. R., (1957), "Networks of Waiting Lines", Oper. Res., 6, 518-521.

10. Kleinrock, J. R., (1977), Queueing Systems, Volume I Theory, John Wiley and Sons, New York, New York.

11. Lindgren, B. W. and McElrath, G. W., (1970), Introduction to Probability and Statistics, 3rd edition, The Macmillan Co., New York, New York.

12. Melamed, B., (1976), "Analysis and Simplifications of Discrete Event Systems and Jackson Queueing Networks", Tech report 76-6, Dept. of I.O.E., University of Michigan, Ann Arbor, Michigan.

13. Melamed, B., (1979), "Sojourn Times in Queueing Networks", Dept. of I.E. and Mgmt. Sci., Northwestern University, Evanston, Illinois.

14. Reich, E., (1956), "Waiting Times When Queues are in Tandem", Ann. Math. Stat., 28, 768-773.

15. Reich, E., (1963), "Note on Queues in Tandem", Ann. Math. Stat., 34, 338-341.

16. Simon, B. and Foley, R. D., (1979), "Some Results on Sojourn Times in Acyclic Jackson Networks", Mgmt. Sci., 25, 1027-1034.

17. Snedecor, G. W. and Cochran, W. G., (1967), Statistical Methods, 6th edition, The Iowa State University Press, Iowa.

APPENDIX A1

THE COMPUTER PROGRAM

A1.0. Introduction.

In this appendix the simulation program is described. In section (A1.1) a brief discription of the main program and each subprogram is given. Section (A1.2) contains flowcharts of the programs. Section (A1.3) contains source listings of the programs.

A1.1. Discription of Computer Routines.

1. Main Program.

The main program calls the subroutines used in the simulation.

2. Subroutine PAR1 and PAR2.

These two subroutines read in the parameters. The parameters include $(p,\lambda,\mu_1,\mu_2,\mu_3)$, the number of iterations, the seed numbers plus the dimensions of many of the vectors.

3. Subroutine INTL.

This simulation initializes the simulation's variables. The queue lengths are initialized by the use of a geometric random number generator, section (4.2). The tagged customer is added to the first queue and his position in the queue recorded. The clock and the sojourn time variables are initialized to zero.

4. Subroutine CHECK.

This subroutine checks the queue lengths of the three queues. For $i = 1,2,3$, if the queue length at Q_i is zero, then service rate at Q_i is set equal to zero. Otherwise Q_i's service rate is set to equal μ_i.

5. Subroutine NEXT.

This subroutine generates the next event. Also it updates the queue length vector, as well as, the queue and the position in the queue of the tagged customer.

The time and the type of the next event are determined by two random generators. The time of the next event is determined by exponential random number generator. This time is added to the clock. An uniform random is generated to determine the type of the next event. Depending on the interval in which the number lies the type of the next event is determined, section (4.3).

A subroutine depending on the type of the next event is called to update the queue length vector, as well as, the queue and the position in the queue of the tagged customer. The subroutines and their functions are listed below.

5.1. Subroutine ARR.

This subroutine is called when the next event is an arrival to Q_1. This routine adds one to the queue length at Q_1.

5.2. Subroutine DEPAB.

This subroutine is called when the next event is a departure from Q_1 who goes to Q_2. First, one is subtracted from Q_1 and one is added to Q_2. If the tagged customer is in Q_1, but not in service there, his position in the queue is moved forward one. If he is in service at Q_1, then he moves to the last position in Q_2. His sojourn time at Q_1 is set equal to clock.

5.3. Subroutine DEPAC.

This subroutine is called when the next event is a departure from

Q_1 who goes to Q_3, provided he is not the tagged customer. In this case, one is deleted from Q_1 and one is added to Q_3. If the tagged customer is in Q_2, but not in service there, his position is moved forward one. If the departure was the tagged customer, one is deleted from Q_1, one is added to Q_2, and the tagged customer becomes the last customer in Q_2. His sojourn time is set equal to clock.

5.4. Subroutine DEPBC.

This subroutine is called when the next event is a departure from Q_2 who goes to Q_3. One is deleted from Q_2 and one is added to Q_3. If the tagged customer is in Q_2, but not in service there, his position in Q_2 is moved forward one. If he is in service there, he moves to the end of Q_3 and his sojourn time in Q_2 is calculated as clock minus his sojourn time in Q_1.

5.5. Subroutine DEPC.

If the next event is a departure from Q_3, this subroutine is called. One is subtracted from Q_3. If the tagged customer is in Q_3, but not in service, his position in Q_3 is moved up one. If the tagged customer is in service at Q_3, his sojourn time at Q_3 is calculated as clock minus the sum of his sojourn times in Q_1 and Q_2. The total sojourn time is calculated as clock. An iteration of the simulation is completed.

6. Subroutine STAT.

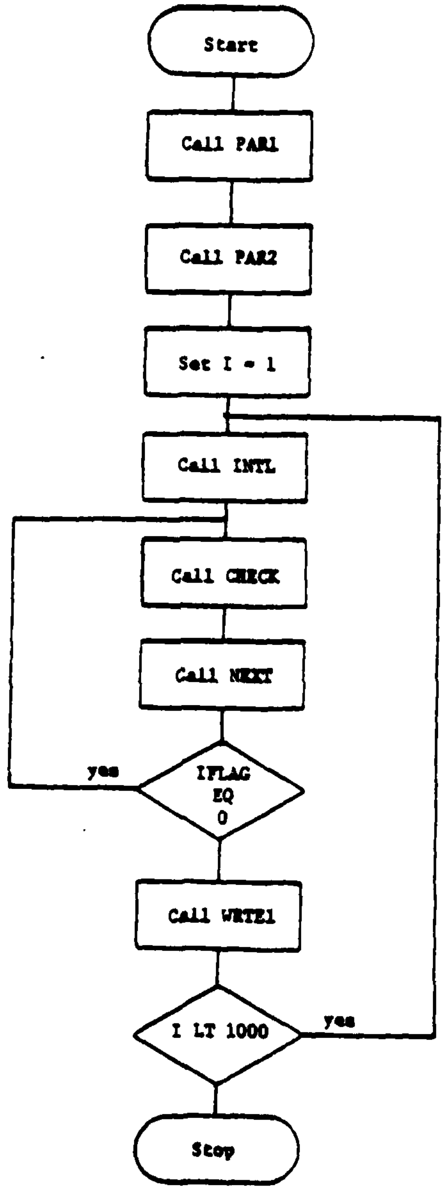
This subroutine calls the IMSL routine, BECORI, which determines the expected sojourn time and the variance in each queue. It also calculated the matrix of correlation coefficients.

7. Subroutines GRPH1 and GRPH2.

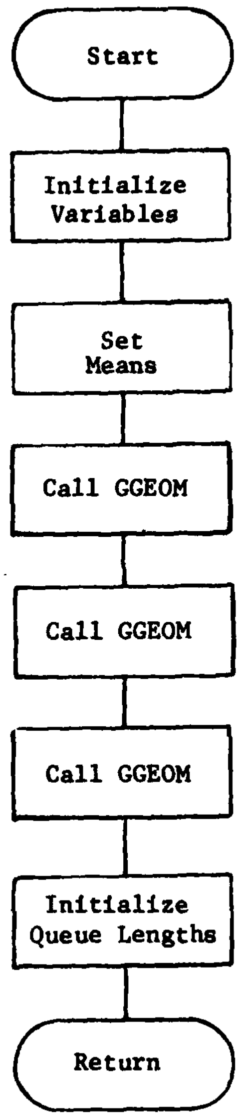
GRPH1 plots the logs of the tagged customer's sojourn times in Q_1

versus the logs of their sojourn time in Q_3. GRPH2 plots the sample distribution versus the distribution assuming independence, section (5.2).

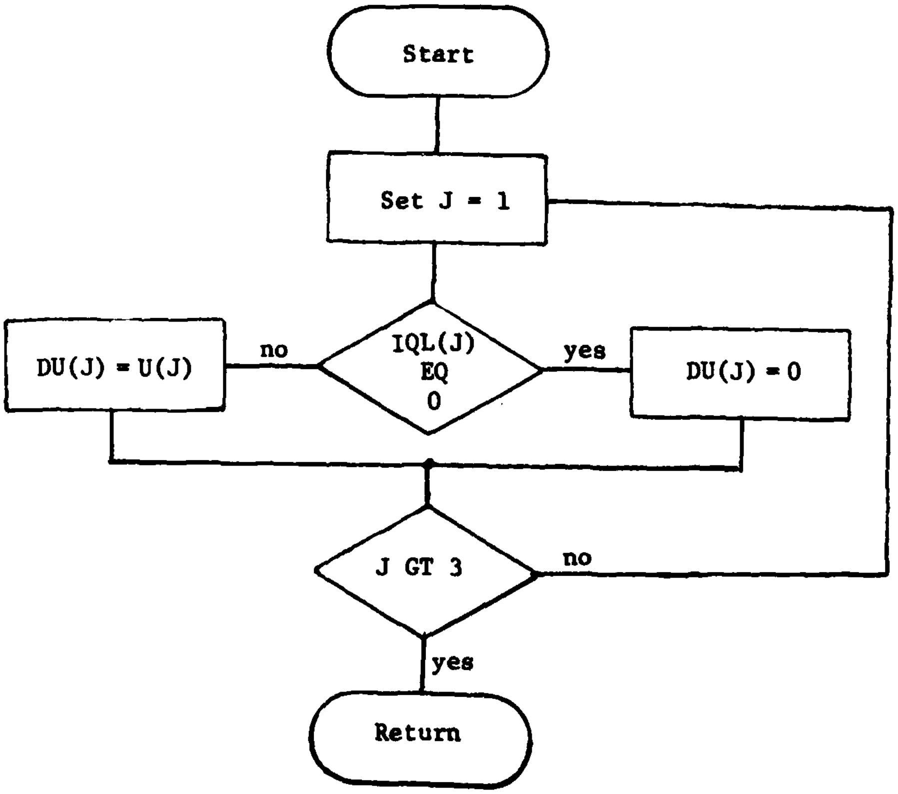
A1.2. Flowcharts.



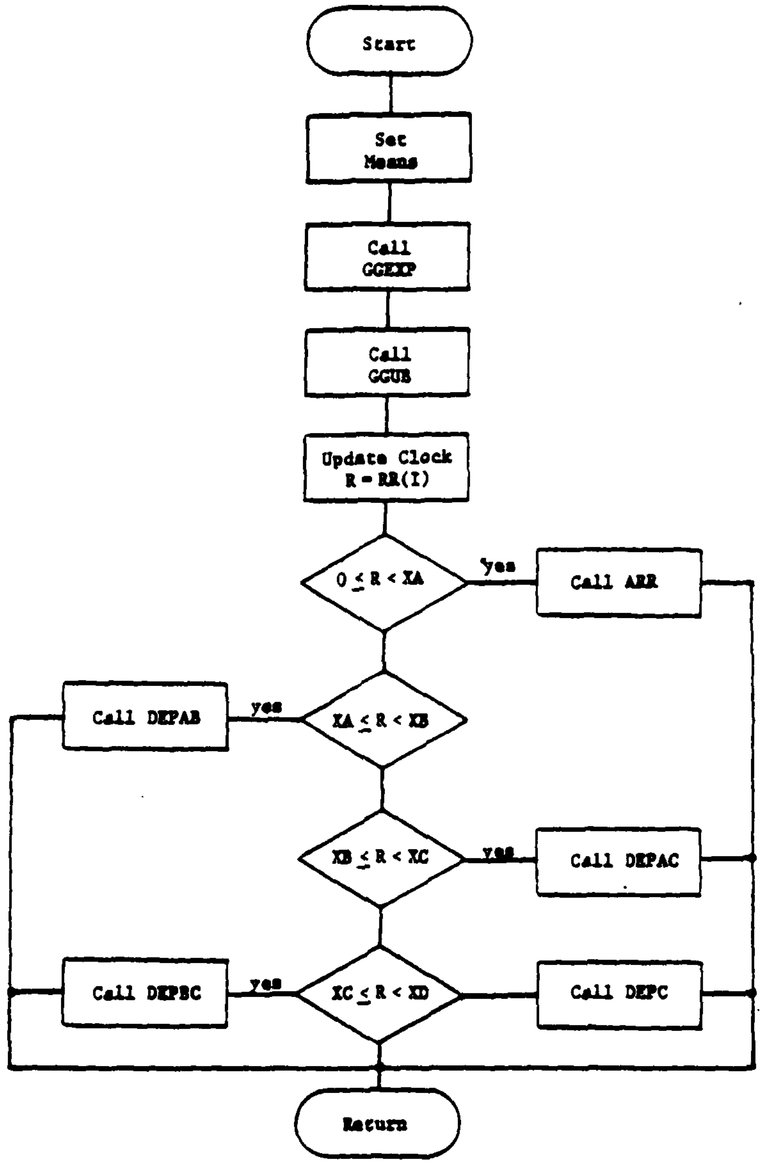
Main Program



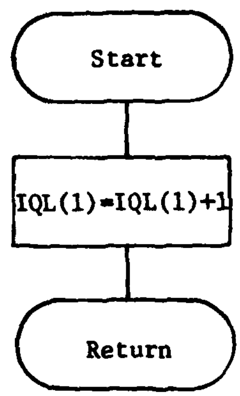
Subroutine INTL



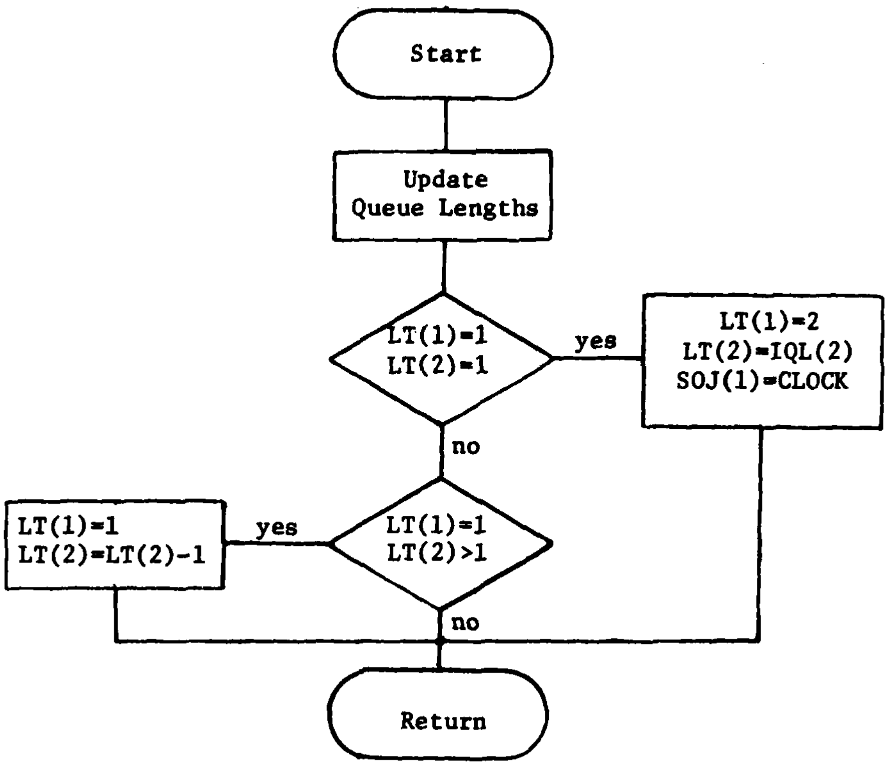
Subroutine CHECK



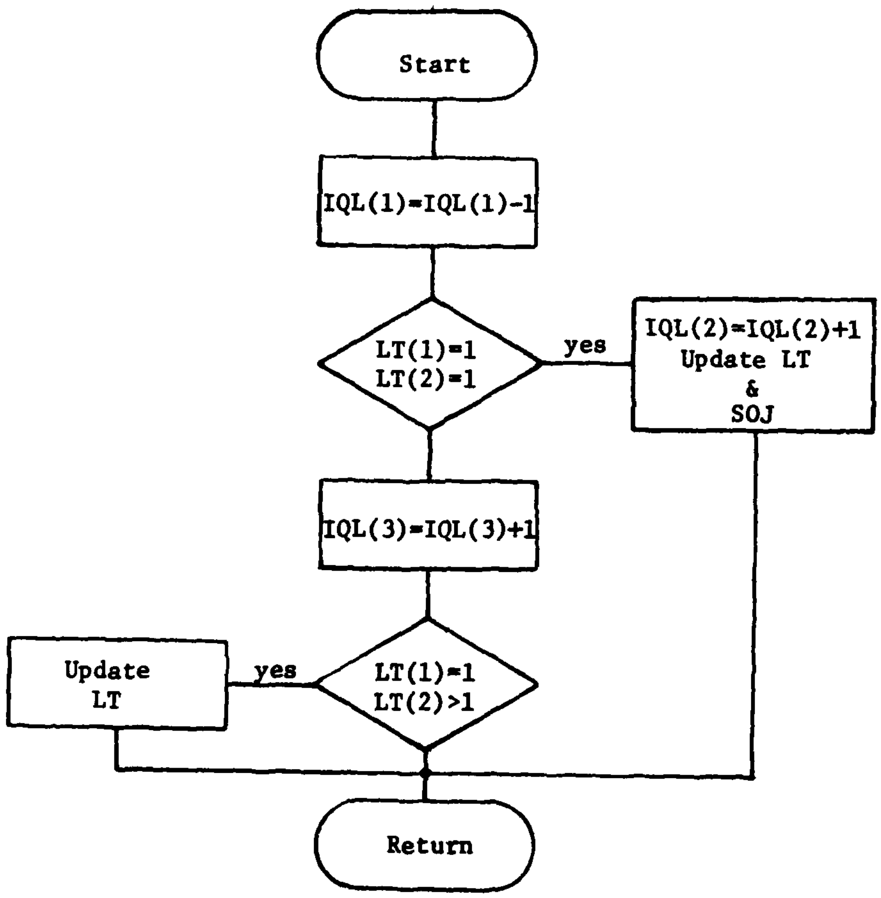
Subroutine NEXT



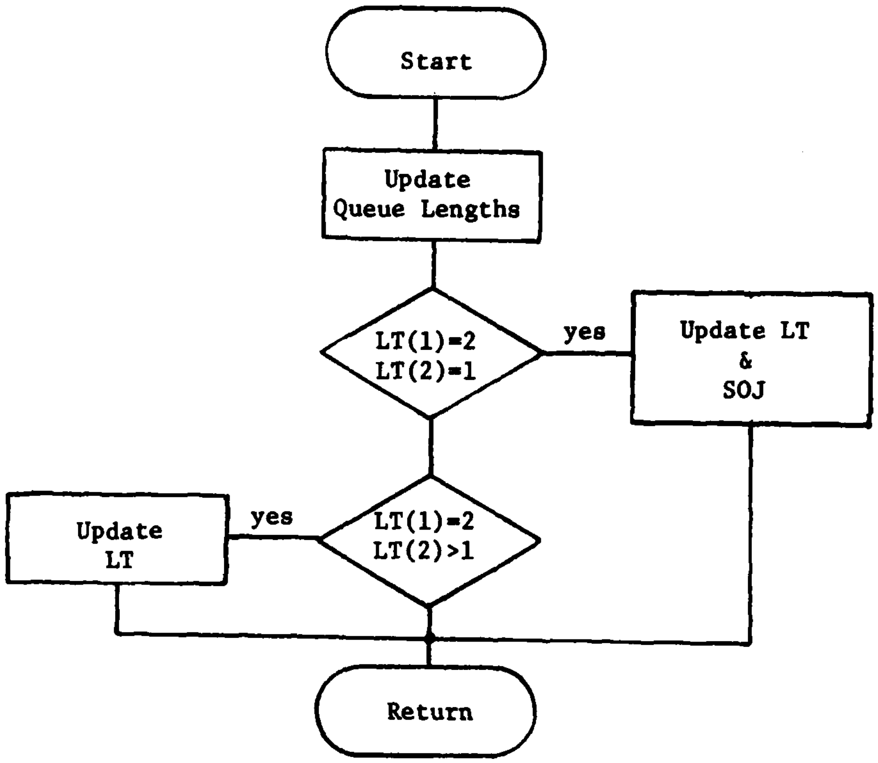
Subroutine ARR



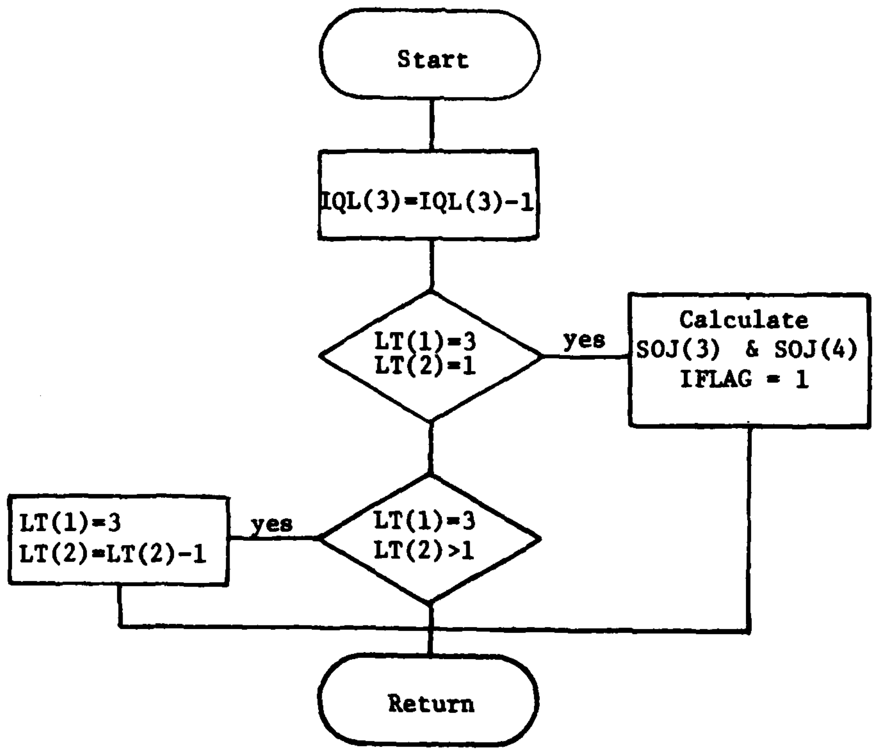
Subroutine DEPAB



Subroutine DEPAC



Subroutine DEPBC



Subroutine DEPC

A1.3. <u>Program Listings</u>.

1. Definition of Variables.

1.1. Network Parameters.

P - the switching probability.

A - the arrival rate to Q_1.

U - a three dimensional vector where U(i), i = 1,2,3, is the service rate at Q_i.

DU - a three dimensional vector where DU(i), i = 1,2,3, is either 0 or U(i) depending on whether Q(i) is empty or not.

1.2. Seed Numbers.

ISA - seed for generating initial queue length at Q_1.

ISB - seed for generating initial queue length at Q_2.

ISC - seed for generating initial queue length at Q_3.

ISD - seed for generating the time of the next event.

ISE - seed for generating the type of the next event.

1.3. Network Variables.

IQL - a three dimensional vector, where IQL(i), i = 1,2,3, is the queue length at Q_i.

LT - a two dimensional vector, where LT(1) is the queue of the tagged customer, and LT(2) is the tagged customer's position in the queue.

SOJ - a three dimension vector, where SOS(i), i = 1,2,3, is the tagged customer's sojourn time in Q_i.

TSOJ - the tagged customer's total sojourn time.

CLOCK - running time of the customer's sojourn time in the network.

IFLAG - tells the simulation when the last tagged customer leaves the network by going from zero to one.

1.4. Statistical Variables.

Z - a (1000 x 4) dimension array where Z(i,j), i = 1,$\cdots$,1000, is the ith tagged customer's total sojourn time.

XM - 4 dimensional vector where XM(i) is the expected value of Z($\cdot$,i), i = 1,$\cdots$,4.

S - 4 dimension vector where S(i) is the variance of Z($\cdot$,i), i = 1,$\cdots$,4.

2. Program Listings.

On the following pages are listings of the computer routines.

```
      EXTERNAL BECORI,PLOT,PLOTS,SYMBOL
      REAL*8 ISA,ISB,ISC,ISD,ISE
      DIMENSION IRA(2),IRB(2),IRC(2),WK(2),IQL(3),SOJ(4),LT(2),DU(3),
     H       U(3),TSOJ(3),ESOJ(3),
     H  Z(1000,4),XM(4),S(4),R(10),AX(10)
C THE MAIN PROGRAM CALLS THE SUBROUTINES WHICH ARE USED IN THE SIMULATION
      CALL PAR1(NI,N,NZ,NM,NU,NN,MM,AX)
C THERE ARE NI SETS OF DATA IN THE SIMULATION
      DO 8 II = 1,NI
      CALL PAR2(P,A,U,ISA,ISB,ISC,ISD,ISE,TSOJ)
      CALL WRTEP(P,A,U,NN,ISA,ISB,ISC,ISD,ISE)
C FOR EACH SET OF PARAMETER SETTINGS THE SIMULATION IS RUN N TIMES
      DO 15 I = 1,N
      CALL INITL(IQL,SOJ,LT,DU,A,IFLAG,CLOCK,P,NN,MM,U,ISA,ISB,ISC)
   11 CONTINUE
      CALL CHECK(IQL,U,DU,NN)
      CALL NEXT(IQL,SOJ,LT,DU,A,IFLAG,CLOCK,P,NN,MM,ISD,ISE,TSOJ)
      IF (IFLAG.EQ.0)  GO TO 11
      CALL WRTE1(SOJ,NM)
C IF THE NUMBER OF ITERATIONS IS LESS THAN N THEN WE GO BACK AND RUN ANOTHER
C ITERATION OTHERWISE THE STATISTICS ARE DONE ON THE DATA
   15 CONTINUE
    8 CONTINUE
      REWIND 10
      DO 7 II = 1,NI
      CALL READAT(Z,N,NM)
      CALL STAT(Z,XM,S,R,N,NM,NU)
      CALL TEST1(Z,N,NM,XM)
    7 CONTINUE
      REWIND 10
      DO 3 II = 1,NI
      CALL READAT(Z,N,NM)
```

```
      CALL GRPH(Z,N,NN,NI,AX,NZ)
    3 CONTINUE
      STOP
      END
      SUBROUTINE PAR1(NI,N,NZ,NM,NU,NN,MM,AX)
      DIMENSION AX(10)
C SUBROUTINE PAR1 READS ,N, THE NUMBER OF ITERATIONS THE SIMULATION
C REQUIRES. IT ALSO READS NI THE NUMBER DIFFERENT PARAMETER SETTINGS IN
C THE SIMULATION. IT READS N2,NM,NU,NN,MM,THE DIMENSIONS OF THE VECTORS
C USED IN THE SIMULATION. IT READS AX(10) THE AXIS OF THE PLOT OF THE LOGS
C OF THE CORRELATION COEFFICIENTS
      READ(5,97) NI,N,NZ,NM,NU,NN,MM
   97 FORMAT(I2,I5,I2,I1,I1,I1,I1)
      READ(5,50) (AX(I),I=1,10)
   50 FORMAT(10(F3.1))
      RETURN
      END
      SUBROUTINE PAR2(P,A,U,ISA,ISB,ISC,ISD,ISE,TSOJ)
      REAL*8 ISA,ISB,ISC,ISD,ISE
      DIMENSION U(3),TSOJ(3)
C SUBROUTINE PAR2 READS THE PARAMETER SETTINGS OF THE NETWORK
C THE ARRIVAL RATE TO Q1,A,THE SWITCHING PARAMETER,P,AND THE SERVICE
C RATES AT QI,I=1,2,3,U(I)
C IT READS IN THE SEED NUMBERS FOR THE RANDOM NUMBER GENERATORS,
C ISA,ISB,ISC.ISD,ISE. IT ALSO INTIALIZES THE TOTAL SOJOURN TIME
C AT EACH QUEUE TC 0
      READ(5,100) P,A,(U(J),J=1,3)
  100 FORMAT(5(2X,F5.2))
      READ(5,98) ISA,ISB,ISC,ISD,ISE
   98 FORMAT(5(D10.0,5X))
      DO 9 K=1,3
    9 TSOJ(K) = 0.
```

```
      RETURN
      END
      SUBROUTINE WRTEP(P,A,U,NN,ISA,ISB,ISC,ISD,ISE)
      REAL*8 ISA,ISB,ISC,ISD,ISE
      DIMENSION U(3)
C SUBROUTINE WRTEP PRINTS OUT THE PARAMETER SETTINGS AND THE SEED NUMBERS
C SUBROUTINE WRTE1 WRITES EACH TAGGED CUSTOMERS SOJOURN TIME AT EACH QUEUE
C AND HIS TOTAL SOJOUJN TIME ONTO A DISK
      WRITE(6,500)
      WRITE(6,501) P
      WRITE(6,502) A
      WRITE(6,503) U(1)
      WRITE(6,504) U(2)
      WRITE(6,505) U(3)
      WRITE(6,506)
      WRITE(6,507)
      WRITE(6,508) ISA,ISB,ISC,ISD,ISE
  500 FORMAT('1',35X,'PARAMETER SETTINGS')
  501 FORMAT(////,20X,'SWITCHIMG PROBABILITY',12X,F5.2)
  502 FORMAT(////,20X,'ARRIVAL RATE TO QUEUE 1',10X,F5.2)
  503 FORMAT(////,20X,'SERVICE RATE AT QUEUE 1',10X,F5.2)
  504 FORMAT(////,20X,'SERVICE RATE AT QUEUE 2',10X,F5.2)
  505 FORMAT(////,20X,'SERVICE RATE AT QUEUE 3',10X,F5.2)
  506 FORMAT(////,39X,'THE SEED NUMBERS')
  507 FORMAT(///,10X,'ISA',15X,'ISB',15X,'ISC',15X,'ISD',15X,'ISE')
  508 FORMAT(//,5(5X,D13.7))
      RETURN
      END
      SUBROUTINE WRTE1(SOJ,NM)
      DIMENSION SOJ(4)
      SOJ(4) = SOJ(1) + SOJ(2) + SOJ(3)
      WRITE(10,301) (SOJ(K),K=1,4)
```

```
  301 FORMAT(4(E13.6))
      RETURN
      END
      SUBROUTINE NEXT(IQL,SOJ,LT,DU,A,IFLAG,CLCCK,P,NN,MM,ISD,ISE,TSOJ)
      REAL*8 ISD,ISE
      DIMENSION TI(2),IQL(3),SOJ(4),LT(2),DU(3),TSOJ(3),RR(2)
C SUBROUTINE NEXT DETERMINES THE TIME AND THE TYPE OF THE NEXT EVENT
      N=2
C
C THE MEAN Z OF THE EXPCNENTIAL DISTRIBUTION DETERMINING THE TIME BETWEEN
C EVENTS IS DETERMINED
C
      Z = A + DU(1) + DU(2) + DU(3)
      XA = A/Z
      XB = (A + P*DU(1))/Z
      XC = (A + DU(1))/Z
      XD = (A + DU(1) + DU(2))/Z
      XM = 1./Z
C
C THE EXPONENTIAL AND UNIFORM RANDOM GENERATORS ARE CALLED
C
      CALL GGEXN(ISD,XM,N,TI)
      CALL GGUBS(ISE,N,RR)
C
C THE CLOCK IS UPDATED AND THE TYPE OF THE NEXT EVENT IS DETERMINED
C
      CLOCK = CLOCK + TI(2)
C
C THE INTERVALS WHICH WIL  CCL DETERMINE THE TYPE CF THE NEXT EVENT ARE
C CALCULATED
C DEPENDING ON THE INTERVAL IN WHICH R LIES
C DEPENDING ON THE TYPE OF THE NEXT EVENT ONE OF THE FOLLOWING
```

```
C SUBROUTINES IS CALLED
C SUBROUTINE ARR
C SUBROUTINE DEPAB
C SUBROUTINE DEPAC
C SUBROUTINE DEPBC
C SUBROUTINE DEPC
C
      R = RR(2)
      IF (R.GE.O..AND.R.LT.XA)  GO TO 41
      IF (R.GE.XA.AND.R.LT.XB) GO TO 42
      IF (R.GE.XB.AND.R.LT.XC) GO TO 43
      IF (R.GE.XC.AND.R.LT.XD) GO TO 44
      CALL DEPC(IQL,LT,SOJ,CLOCK,NN,MM,IFLAG,TSCJ)
      RETURN
   41 CONTINUE
      CALL ARR(IQL,NN)
      RETURN
   42 CONTINUE
      CALL DEPAB(IQL,LT,SCJ,CLOCK,NN,MM,TSCJ)
      RETURN
   43 CONTINUE
      CALL DEPAC(IQL,LT,SOJ,CLOCK,NN,MM,TSCJ)
      RETURN
   44 CONTINUE
      CALL DEPBC(IQL,LT,SOJ,CLOCK,NN,MM,TSOJ)
      RETURN
      END
      SUBROUTINE INITL(IQL,SCJ,LT,DU,A,IFLAG,CLCCK,P,NN,MM,U,
     H                    ISA,ISB,ISC)
      REAL*8 ISA,ISB,ISC
C SUBROUTINE INTL INTIALIZES THE VARIABLES USED IN THE SIMULATION
      DIMENSION IRA(2),IRB(2),IRC(2),WK(2),ICL(3),SOJ(4),LT(2),DU(3),
```

```
     H       U(3)
C
C THE FLAG AND THE CLOCK ARE SET EQUAL TC 0
C
      IFLAG = 0
      NR = 2
      CLOCK = 0.
C
C FOR J=1,2,3,DU(J) AND SCJ(J) ARE SET EQUAL TC 0 AS WELL AS SOJ(4)
C
      DC 30 J = 1,3
      DU(J) = U(J)
      SOJ(J) = 0.
   30 CONTINUE
      SOJ(4) = 0.
C
C THE MEANS OF THE GEOMETRIC RANDCM NUMBER GENERATORS ARE SET
C
      PA = 1.- (A/DU(1))
      PB = 1.- (P*A/DU(2))
      PC = 1.- (A/DU(3))
C
C THE RANDOM GENERATORS ARE CALLEC AND THE QUEUE LENGTHS INITIALIZED
C
      CALL GGEOT(ISA,NR,PA,WK,IRA)
      CALL GGEOT(ISB,NR,PB,WK,IRB)
      CALL GGEOT(ISC,NR,PC,WK,IRC)
      IQL(1) = IRA(2)
      IQL(2) = IRB(2) -1
      IQL(3) = IRC(2) - 1
      LT(1) = 1
      LT(2) =IQL(1)
```

```
      RETURN
      END
      SUBROUTINE CHECK(IQL,U,DU,NN)
      DIMENSION IQL(3),U(3),DU(3)
C
C SUBROUTIMNE CHECK CHECKS THE QUEUE LENGTH AT EACH QUEUE. IF THE QUEUE
C LENGTH AT QI,I=1,2,3,THEN DU(I) IS SET TO 0,OTHERWISE IT IS SET AT U(I)
C
      DO 20 J = 1,3
      IF(IQL(J).EQ.0) GO TO 21
      DU(J) = U(J)
      GO TO 20
   21 DU(J) = 0
   20 CONTINUE
      RETURN
      END
      SUBROUTINE ARR(IQL,NN)
      DIMENSION IQL(3)
C
C
C ONE IS ADDEC TC THE QUEUE LENGTH AT Q1
C SUBROUTINE ARR IS CALLED WHEN THE NEXT EVENT IS AN ARRIVAL TO Q1
      IQL(1) = IQL(1) + 1
      RETURN
      END
      SUBROUTINE DEPAB(IQL,LT,SOJ,CLOCK,NN,MM,TSOJ)
      DIMENSION IQL(3),LT(2),SOJ(4),TSOJ(3)
C
C SUBROUTINE DEPAB IS CALLED WHEN THE NEXT EVENT IS A DEPARTURE FROM
C Q1 WHO GOES TO Q2
C ONE IS SUBTRACTED FROM Q1 AND ONE IS ADDED TO Q2
C
```

```
      IQL(1) =IQL(1) - 1
      IQL(2) =IQL(2) + 1
C
C THE QUEUE AND POSITION OF THE TAGGED CUSTOMER ARE CHECKED
C
C
C IF THE TAGGED CUSTOMER IS IN THE Q1 BUT IN SERVICE THERE
C HIS POSITION Q1 IS MOVED UP 1
C IF HE IS IN SERVICE AT Q1 THEN HE MOVES TO THE LAST POSTION IN Q2
C
      IF (LT(1).EQ.1.AND.LT(2).EQ.1) GO TO 81
      IF (LT(1).EQ.1.AND.LT(2).GT.1) GO TO 82
      RETURN
   82 LT(1) =1
      LT(2) = LT(2) -1
      RETURN
   81 LT(1) =2
      LT(2) = IQL(2)
C
C THE TAGGED CUSTOMERS SOJOURN TIME IN Q1 IS RECORED
C
      SOJ(1) = CLOCK
      TSOJ(1) = TSOJ(1) + SOJ(1)
      RETURN
      END
      SUBROUTINE DEPAC(IQL,LT,SOJ,CLOCK,NN,MM,TSOJ)
      DIMENSION IQL(3),LT(2),SOJ(4),TSOJ(3)
C
C SUBROUTINE DEPAC IS CALLED WHEN THE NEXT EVENT IS A DEPARTURE FROM Q1
C ONE IS SUBTRACTED FROM Q1
C
      IQL(1) =IQL(1) - 1
```

AD-A093 707 VIRGINIA POLYTECHNIC INST AND STATE UNIV BLACKSBURG --ETC F/G 12/1
A SIMULATION ANALYSIS OF SOJOURN TIMES IN A JACKSON NETWORK.(U)
DEC 80 P C KIESSLER N00014-77-C-0743
UNCLASSIFIED VTR-80-16 NL

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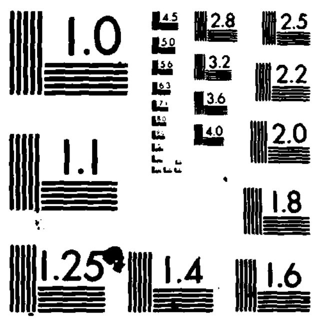

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

```
C
C THE QUEUE AND POSITION OF THE TAGGED CUSTOMER ARE CHECKED
C IF HE IS IN SERVICE AT Q1 THEN ONE IS ADDED TO Q2 AND THE TAGGED CUSTOMER
C IS MOVED TO THE LAST POSITION OF Q2
C AND HIS SOJOURN TIME IS RECORDED
C IF THE TAGGED CUSTOMER IS NOT IN SERVICE AT Q1 THEN ONE IS ADDED TO Q3
C IF HE IS IN Q1 HIS POSITION IN Q1 IS MOVED UP 1
C
      IF (LT(1).EQ.1.AND.LT(2).EQ.1) GO TO 51
      IQL(3) =IQL(3) + 1
      IF (LT(1).EQ.1.AND.LT(2).GT.1) GO TO 52
      RETURN
   52 LT(1) = 1
      LT(2) = LT(2) -1
      RETURN
   51 CONTINUE
      IQL(2) =IQL(2) + 1
      LT(1) = 2
      LT(2) =IQL(2)
      SOJ(1) = CLOCK
      TSOJ(1) = TSOJ(1) + SOJ(1)
      RETURN
      END
      SUBROUTINE DEPBC(IQL,LT,SOJ,CLOCK,NN,MM,TSOJ)
      DIMENSION IQL(3),LT(2),SOJ(4),TSOJ(3)
C DEPBC IS CALLED WHEN THE NEXT EVENT IS A DEPARTURE FROM Q2 WHO GOES TO
C Q3
C ONE IS DELETED FROM Q2 AND ONE IS ADDED TO Q2
C ONE IS SUBTRACTED FROM Q2 AND ONE IS ADDED TO Q3
C
      IQL(2) =IQL(2) - 1
      IQL(3) =IQL(3) + 1
```

```
C
C THE QUEUE AND PCSITION CF THE TAGGEC CUSTCMER ARE CHECKED
C IF HE IS IN Q2 BUT NCT IN SERVICE HIS POSITICN IS MOVED UP 1
C IF HE IS IN SERVICE AT G2 HE MOVES TO THE LAST POSITION IN Q3
C AND HIS SOJOURN TIME IN Q2 AND HIS PRESENT TOTAL SOJOURN TIMES ARE
C RECORDED
C
      IF (LT(1).EC.2.AND.LT(2).EQ.1) GO TO 61
      IF (LT(1).EQ.2.AND.LT(2).GT.1) GO TC 62
      RETURN
   62 LT(1) = 2
      LT(2) = LT(2) - 1
      RETURN
   61 LT(1) = 3
      LT(2) =IQL(3)
      SOJ(2) = CLOCK - SCJ(1)
      TSOJ(2) = TSOJ(2) + SOJ(2)
      RETURN
      END
      SUBROUTINE CEPC(IQL,LT,SOJ,CLOCK,NN,MM,IFLAG,TSOJ)
      DIMENSION IQL(3),LT(2),SOJ(4),TSCJ(3)
C
C DEPC IS CALLEC WHEN THE NEXT EVENT IS ADEPARTURE FROM Q3
C ONE IS SUBTRACTED FROM Q3
C THE QUEUE AND PGSITION OF THE TAGGED CUSTOMER ARE CHECKED
C IF HE IS IN Q3 BUT NOT IN SERVICE THERE HIS PCSITION IN Q3 IS MOVED
C UP 1
C IF HE IS IN SERVICE AT Q3 HE LEAVES THE NETWORK AND HIS SOJOURN TIME
C IN Q3 AND HIS TOTAL SOJOURN TIME IN THE NETWORK ARE RECORDED
C
      IQL(3) =IQL(3) - 1
      IF (LT(1).EQ.3.AND.LT(2).EQ.1) GO TO 71
```

```
      IF (LT(1).EQ.3.AND.LT(2).GT.1) GO TO 72
      RETURN
   72 LT(1) = 3
      LT(2) = LT(2) - 1
      RETURN
   71 SOJ(3) = CLOCK - SOJ(2) - SOJ(1)
      TSOJ(3) = TSOJ(3) + SOJ(3)
      IFLAG = 1
      RETURN
      END
      SUBROUTINE READAT(Z,N,NM)
      DIMENSION Z(1000,4)
      DO 90 I = 1,N
      READ(10,302) (Z(I,J),J=1,4)
  302 FORMAT(4(E13.6))
   90 CONTINUE
      RETURN
      END
      SUBROUTINE STAT(Z,XM,S,R,N,NM,NU)
      DIMENSION Z(1000,4),XM(4),S(4),R(10)
      IX = N
      CALL BECORI(Z,N,NM,IX,XM,S,R,IER)
      WRITE(6,203)
      WRITE(6,201)
      WRITE(6,202) (XM(J),J=1,3)
      WRITE(6,204)
      WRITE(6,201)
      WRITE(6,202) (S(J),J=1,3)
      WRITE(6,205)
      WRITE(6,201)
      WRITE(6,206) R(1)
      WRITE(6,207) R(2),R(3)
```

```
      WRITE(6,208) R(4),R(5),R(6)
      WRITE(6,209) XM(4)
      WRITE(6,210) S(4)
  201 FORMAT('0',23X,'QUEUE 1',23X,'QUEUE 2',23X,'QUEUE 3')
  202 FORMAT(' ',3X,3(17X,E13.6))
  203 FCRMAT('1',44X,'EXPECTED SCJURN TIMES')
  204 FORMAT(////,46X,'STANDARD DEVIATIONS')
  205 FORMAT(////,43X,'CORRELATICN CCEFFICIENTS')
  206 FCRMAT('0','QUEUE 1',13X,E13.6)
  207 FORMAT('0','QUEUE 2',13X,2(E13.6,17X))
  208 FORMAT('0','QUEUE 3',13X,3(E13.6,17X))
  209 FORMAT(////,10X,'TOTAL EXPECTED SOJURN TIME',25X,E13.6)
  210 FCRMAT(////,10X,'TOTAL SOJURN TIME STANCARD DEVIATION',15X,E13.6)
      RETURN
      END
      SUBROUTINE TEST1(Z,N,NM,XM)
      REAL LCL
      DIMENSION Z(1000,4),XM(4)
      XN = FLOAT(N)
      SSX = 0.
      SSY = 0.
      SSXY = 0.
      B = 0.
      S12 = 0.
      TSTAT = 0.
      DC 12 I = 1,N
      SSX = SSX + Z(I,1)**2
      SSXY = SSXY + Z(I,1)*Z(I,3)
      SSY = SSY + Z(I,3)**2
   12 CONTINUE
      XBAR = XM(1)
      YBAR = XM(3)
```

```
      B = SSXY/SSX
      S12 = (SSY-B*SSXY)/(XN-2.)
      S12 = SQRT(S12)
      TSTAT = B/(S12/SQRT(SSX))
      CONS = 1.96*S12/(SQRT(SSX))
      UCL = B + CONS
      LCL = B - CONS
      WRITE(6,400)
      WRITE(6,401) SSX
      WRITE(6,402) SSY
      WRITE(6,403) SSXY
      WRITE(6,404) B
      WRITE(6,405) S12
      WRITE(6,406) TSTAT
      WRITE(6,407) CONS
      WRITE(6,408) LCL,UCL
  400 FORMAT('1',20X,'REGRESSION ANALYSIS')
  401 FORMAT(////,10X,'SSX',25X,E13.6)
  402 FORMAT(////,10X,'SSY',25X,E13.6)
  403 FORMAT(////,10X,'SSXY',24X,E13.6)
  404 FORMAT(////,10X,'B',27X,E13.6)
  405 FORMAT(////,10X,'S12',25X,E13.6)
  406 FORMAT(////,10X,'TSTAT',23X,E13.6)
  407 FORMAT(////,10X,'CONFIDENCE STAT',13X,E13.6)
  408 FORMAT(////,10X,'CONFIDENCE INTERVAL',10X,'(',E13.6,',',E13.6,')')
      RETURN
      END
      SUBROUTINE GRPH(Z,N,NN,NI,AX,NZ)
      DIMENSION Z(1000,4),X(1000),Y(1000),AX(10)
      DO 13 I = 1,N
      DO 12 K = 1,3
      IF (Z(I,K).EQ.0.) GO TO 16
```

```
      W = Z(I,K)
      Z(I,K) = ALOG(W)
      IF (Z(I,K).GT.-7.5) GO TO 14
   16 Z(I,K) = -7.5
   14 CONTINUE
   12 CONTINUE
   13 CONTINUE
      CALL  PLOTS(0,0,50)
      CALL  PLOT(10.0,5.0,-3)
      CALL  PLOT(-5.0,0.0,3)
      CALL  PLOT(5.0,0.0,2)
      CALL  PLOT(0.0,-5.0,3)
      CALL  PLOT(0.0,5.0,2)
      DO 17 I = 1,10
      CALL  SYMBOL(-.025,AX(I),0.05,96,0.,-1)
      CALL  SYMBOL(AX(I),-.025,0.05,16,0.,-1)
   17 CONTINUE
      CALL  SYMBOL(4.0,-0.3,0.15,6HQUEUE1,0.0,6)
      CALL  SYMBOL(-0.1,4.0,0.15,6HQUEUE3,90.,6)
      DO 18 I = 1,N
      X(I) = Z(I,1) * .66
      Y(I) = Z(I,3) * .66
      CALL  PLOT(X(I),Y(I),3)
      CALL  SYMBOL(X(I),Y(I),.01,11,0.0,-1)
   18 CONTINUE
      CALL  PLOT(20.0,0.0,999)
      RETURN
      END
  5 1000104632
     .10   2.00    5.00    5.00    5.00
4955281.D0      8690263.D0      2554945.D0      9761117.D0      2644529.D0
     .10   2.00    5.00    5.00    5.00
```

```
6453531.D0        7931293.D0        6448273.D0        3230585.D0        2194281.D0
     .10    2.00    5.00     5.00     5.00
7424065.D0        4399715.D0        6553649.D0        8399145.D0        2675983.D0
     .10    2.00    5.00     5.00     5.00
1503535.D0        3521641.D0        7194505.D0        2136161.D0        7992425.D0
     .10    2.00    5.00     5.00     5.00
4707549.D0        1215141.D0        6275795.D0        6412665.D0        7215021.D0
```

APPENDIX A2

THE OUTPUT OF THE SIMULATION

A2.1. Listings of the Output.

The following pages consist of the simulations output. There are four tables and 15 graphs. Table 1.1 consists of the seed numbers and the correlation coefficients. Table 1.2 consists of the seed numbers used in the random number generators. Table 1.3 consists of the expected values of the sojourn times in Q_1,Q_2,Q_3, and the total sojourn time. Table 1.4 consists of the standard deviations of the sojourn times in Q_1,Q_2,Q_3, and the total sojourn time. There are five graphs in which the sample distribution and a distribution of the total sojourn time, assuming S_1,S_2, and S_3 are independent, are superimposed. There are five graphs which plot the difference between these two distributions. Finally, there are five scatter plots which plot the log S_1 vs. log S_3. All plots are for the case where the switching parameter, p, is .1.

TABLE 1.1

LIST OF SWITCHING PARAMETERS, p, AND CORRELATION COEFFICIENTS

run #	p	r_1	r_2	r_3
1	0.00	-.0591	-.0260	.0582
2	0.00	-.0237	-.0267	.0530
3	0.00	.0030	-.0421	-.0210
4	0.00	-.0304	-.0193	.0708
5	0.00	-.0187	-.0461	.0341
6	0.01	.0027	-.0219	.0546
7	0.01	-.0019	.0061	.0735
8	0.01	-.0758	-.0436	.0528
9	0.01	-.0273	-.0199	-.0080
10	0.01	-.0435	-.0261	-.0247
11	0.05	-.0189	-.0027	.0599
12	0.05	-.0176	.0023	.0651
13	0.05	.0123	-.0068	.0616
14	0.05	-.0240	.0269	.0085
15	0.05	.0162	.0456	.1034
16	0.10	-.0226	.0062	.0252
17	0.10	.0065	-.0343	.0308
18	0.10	.0295	.0070	.0439
19	0.10	-.0075	-.0368	-.0317
20	0.10	.0357	.0399	-.0029
21	0.25	-.0431	.0236	-.0310
22	0.25	.0288	-.0354	.0023
23	0.25	.0394	-.0169	.0520
24	0.25	.0292	.0054	.0758
25	0.25	.0225	-.0205	.0915
26	0.05	-.0150	.0294	.0252
27	0.05	-.0329	-.0217	-.0182
28	0.05	.0728	-.0216	.0040
29	0.05	-.0092	.0174	.0524
30	0.05	-.0499	.0198	-.0074
31	1.00	-.0432	-.0435	-.0089
32	1.00	.0051	.0168	-.0111
33	1.00	-.0184	-.0144	-.0440
34	1.00	.0206	-.0081	.0403
35	1.00	-.0604	-.0079	.0140

TABLE 1.2

LIST OF SEED NUMBERS FOR THE RANDOM NUMBER GENERATORS

run #	ISA	ISB	ISC	ISD	ISE
1	5686505	6075695	9022909	6656611	7408799
2	3886767	4756481	7698813	6095261	3997223
3	5774081	6769993	8314997	4583415	3793763
4	9459553	3632017	1859429	9434227	3283261
5	7119413	1866371	5532247	7646823	8976881
6	9396535	4884063	2084713	6525565	6646201
7	4934033	6004515	6827285	8503051	5691217
8	7134157	1256655	2635885	6435093	4113511
9	4968463	1798315	8597729	4610481	6765833
10	1505321	3159501	5390073	2217837	1478013
11	4439443	9161075	7560147	9990439	7966695
12	1063411	3370395	5594455	9690947	8042825
13	4250833	3061375	9274763	1829629	9609691
14	1558555	2997589	3515635	3618813	3469395
15	1859363	2855199	2562553	5072091	6784469
16	9396535	4401349	5218021	6118865	5815067
17	4934035	6004517	3001501	7158585	3580603
18	7134157	1256659	2151127	2349565	4655741
19	4968461	1798313	9773585	5185147	5000167
20	9065515	3159501	4944253	5919325	7679713
21	3947547	1662251	7498871	4349879	1795951
22	3699067	9352671	2316749	1740353	8702517
23	4098007	2049239	2379215	2363227	2650441
24	8397429	4153539	5900995	7047717	9820971
25	4694981	6631119	2669459	9243109	6357417
26	2205089	7295829	8592233	2740853	4753903
27	8319733	8394441	4241653	2401025	6133701
28	9932451	3911789	4658309	8390317	6062737
29	3193527	1111339	9254679	5418397	4142617
30	8872083	3333931	6552091	4255979	4397211
31	2669567	9496487	3413679	1538715	5575893
32	6988261	6568043	6992771	6564817	1627501
33	6590997	5266761	3120439	1707527	8778567
34	4005567	5541763	9081617	1229307	6565111
35	2707235	6300393	1497293	2839511	1214349

TABLE 1.3

LIST OF EXPECTED SOJOURN TIMES IN Q_1, Q_2, Q_3, AND TOTAL SOJOURN TIME

run #	ES_1	ES_2	ES_3	ETS
1	.969	.194	.958	2.121
2	.991	.208	.956	2.154
3	1.039	.206	1.022	2.267
4	.982	.201	.955	2.139
5	1.020	.197	.954	2.172
6	1.000	.199	.951	2.150
7	.995	.198	1.022	2.216
8	.978	.199	.930	2.107
9	1.030	.198	.987	2.216
10	1.027	.197	1.026	2.250
11	1.037	.206	.982	2.224
12	1.017	.211	1.041	2.269
13	1.015	.204	.990	2.210
14	1.025	.203	1.057	2.285
15	.978	.208	.988	2.174
16	1.018	.220	.994	2.231
17	.977	.219	1.039	2.235
18	.998	.228	1.002	2.229
19	.969	.215	.951	2.135
20	1.004	.208	1.034	2.246
21	.974	.244	.980	2.120
22	1.023	.259	.972	2.254
23	.994	.237	1.021	2.251
24	1.032	.247	1.007	2.229
25	.956	.233	.948	2.137
26	1.019	.340	1.013	2.373
27	.997	.322	1.057	2.376
28	.992	.322	.937	2.250
29	.976	.337	.959	2.271
30	1.019	.351	.959	2.233
31	.958	.999	1.012	2.969
32	.999	.960	.997	2.957
33	.954	.973	1.024	2.951
34	1.011	1.005	.944	2.960
35	.950	.978	.987	2.915

TABLE 1.4

STANDARD DEVIATIONS OF THE SOJOURN TIMES IN Q_1,Q_2,Q_3, AND THE TOTAL SOJOURN TIME

1	.989	.190	.949	1.411
2	.955	.204	.922	1.371
3	1.008	.190	.971	1.393
4	.979	.193	.973	1.435
5	1.024	.189	.946	1.422
6	.946	.186	.967	1.398
7	1.011	.202	1.035	1.513
8	.940	.199	.918	1.347
9	1.026	.193	.954	1.403
10	1.008	.194	1.016	1.418
11	1.039	.203	.938	1.452
12	.966	.213	1.008	1.455
13	1.079	.204	1.024	1.547
14	1.024	.207	1.003	1.455
15	.944	.202	.979	1.450
16	.991	.216	.953	1.406
17	.991	.217	1.019	1.455
18	.986	.230	1.074	1.512
19	.991	.220	.927	1.347
20	1.001	.227	.969	1.444
21	.988	.242	.977	1.385
22	1.064	.257	.950	1.451
23	.977	.238	1.081	1.516
24	1.037	.242	1.004	1.522
25	.946	.237	.948	1.420
26	1.028	.323	1.023	1.507
27	1.032	.328	1.021	1.463
28	1.042	.313	.952	1.460
29	1.006	.336	.929	1.447
30	1.050	.352	.938	1.438
31	1.001	.971	1.021	1.673
32	.977	.910	1.035	1.678
33	.970	.939	.973	1.620
34	1.082	1.015	.968	1.803
35	.933	.972	.975	1.633

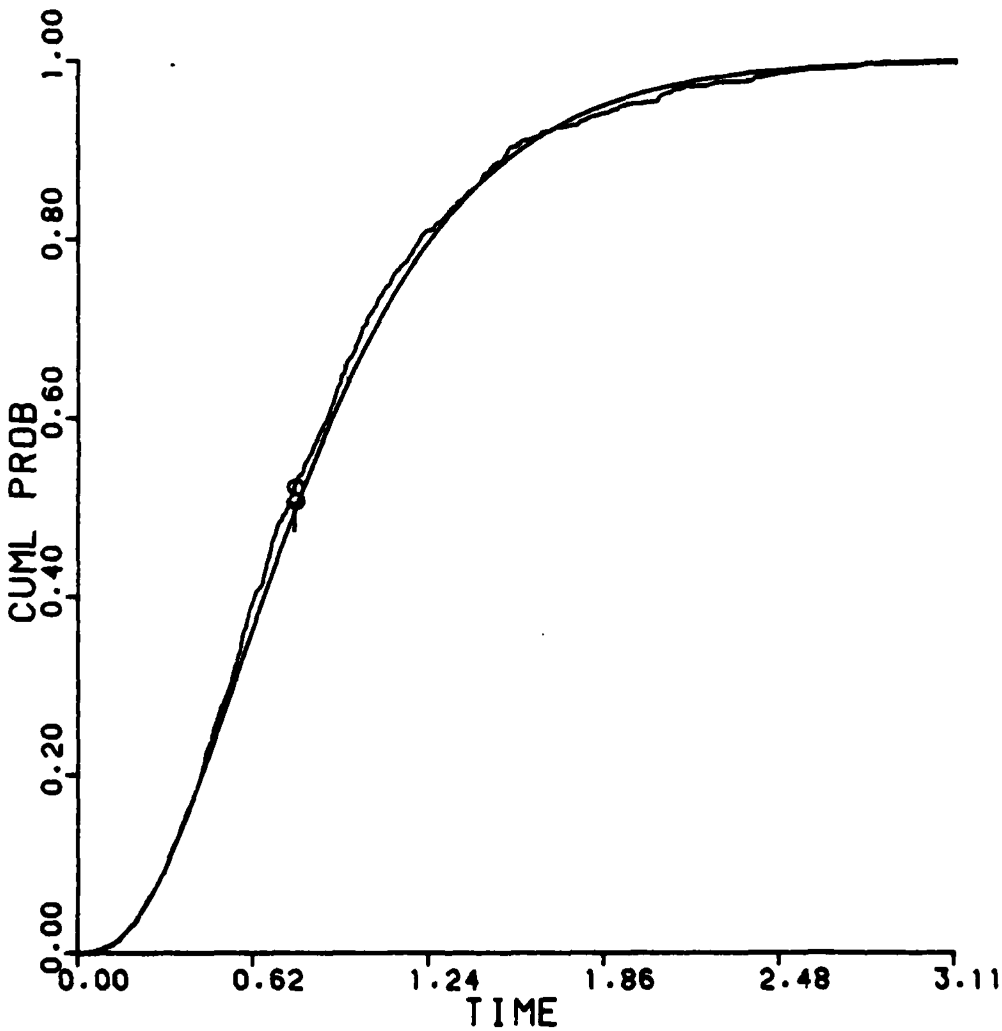

Figure 1.1. A Comparison Between a Sample Distribution, S, and one Assuming Independence, T

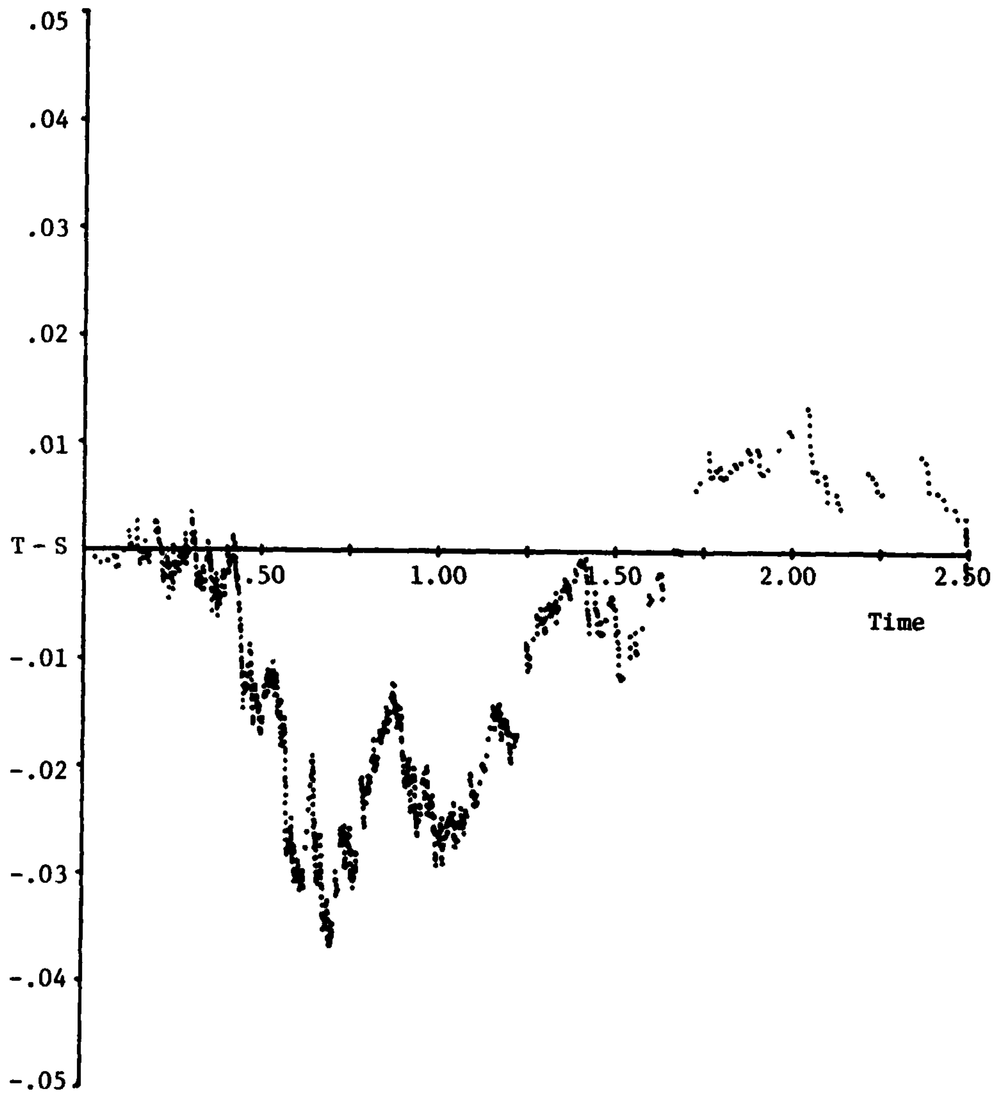

Figure 1.2. The Difference Between T and S

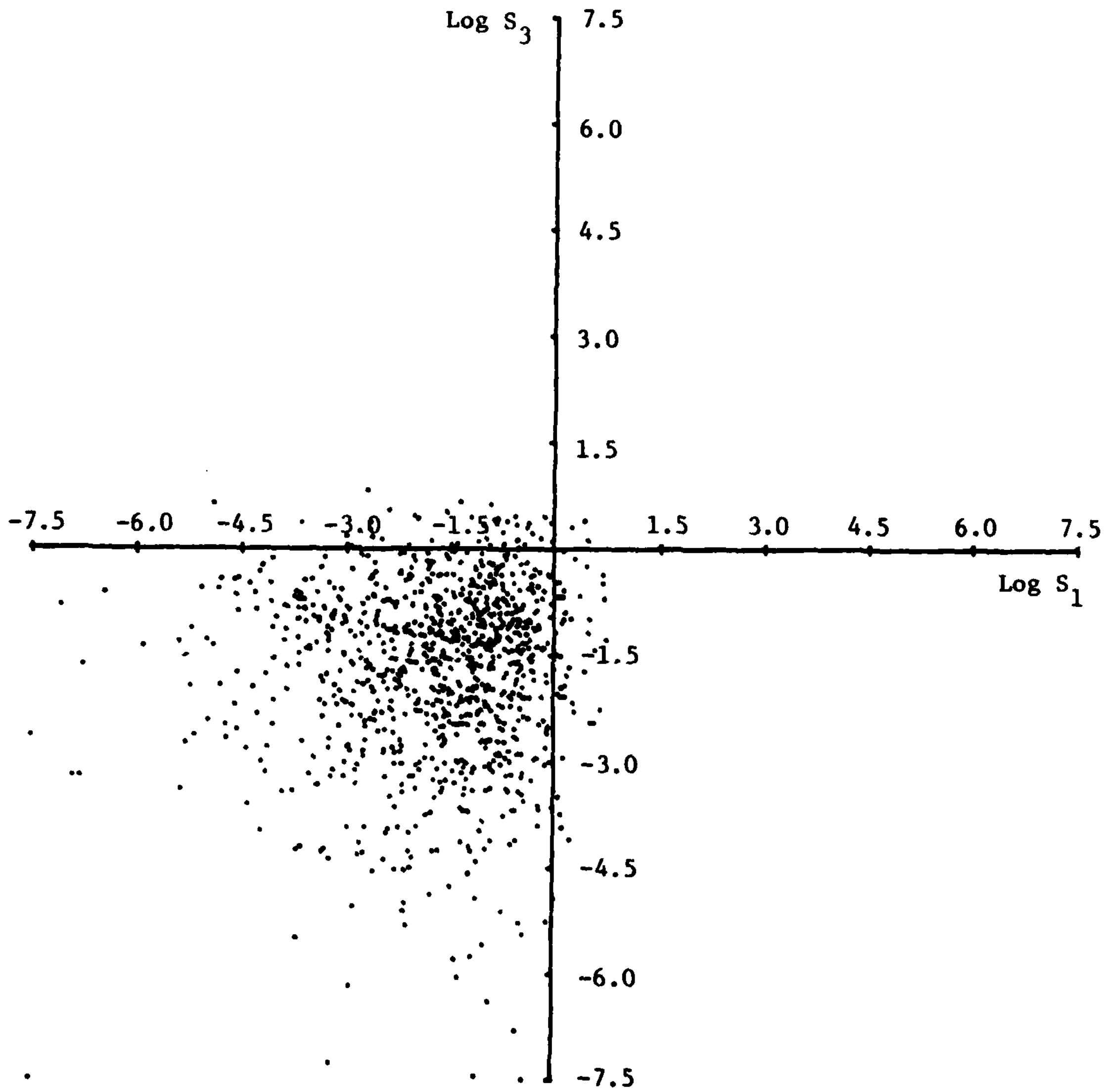

Figure 1.3. Scatter Plot of Log S_1 vs. Log S_3

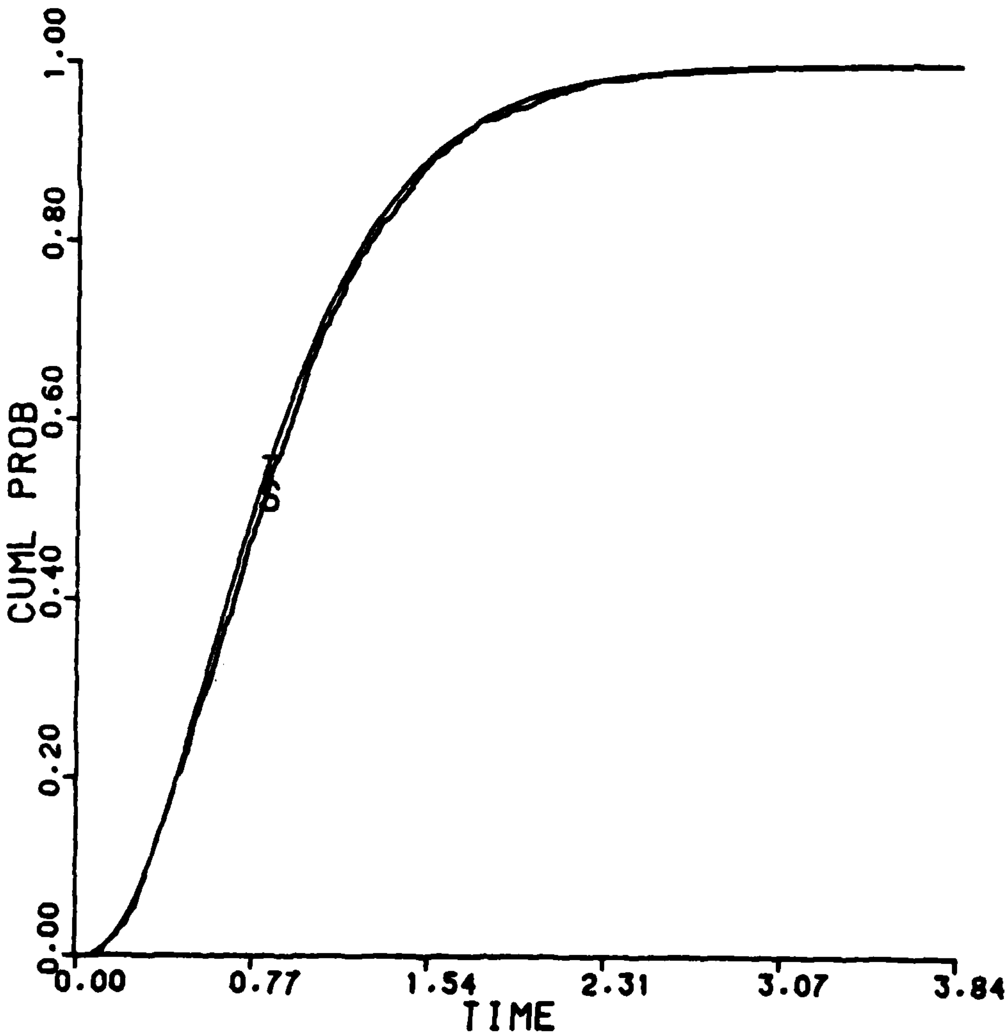

Figure 1.7. A Comparison Between a Sample Distribution, S, and one Assuming Independence, T

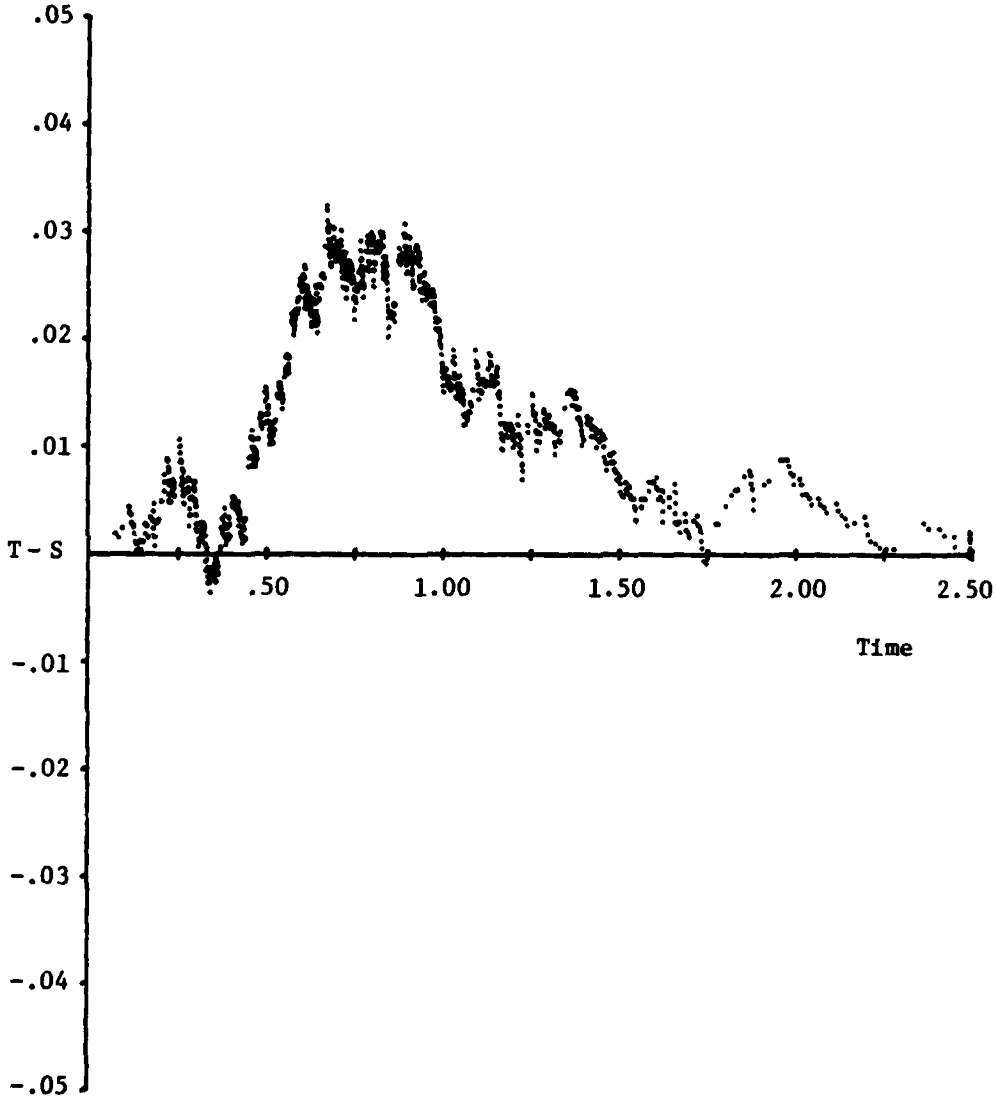

Figure 1.8. The Difference Between T and S

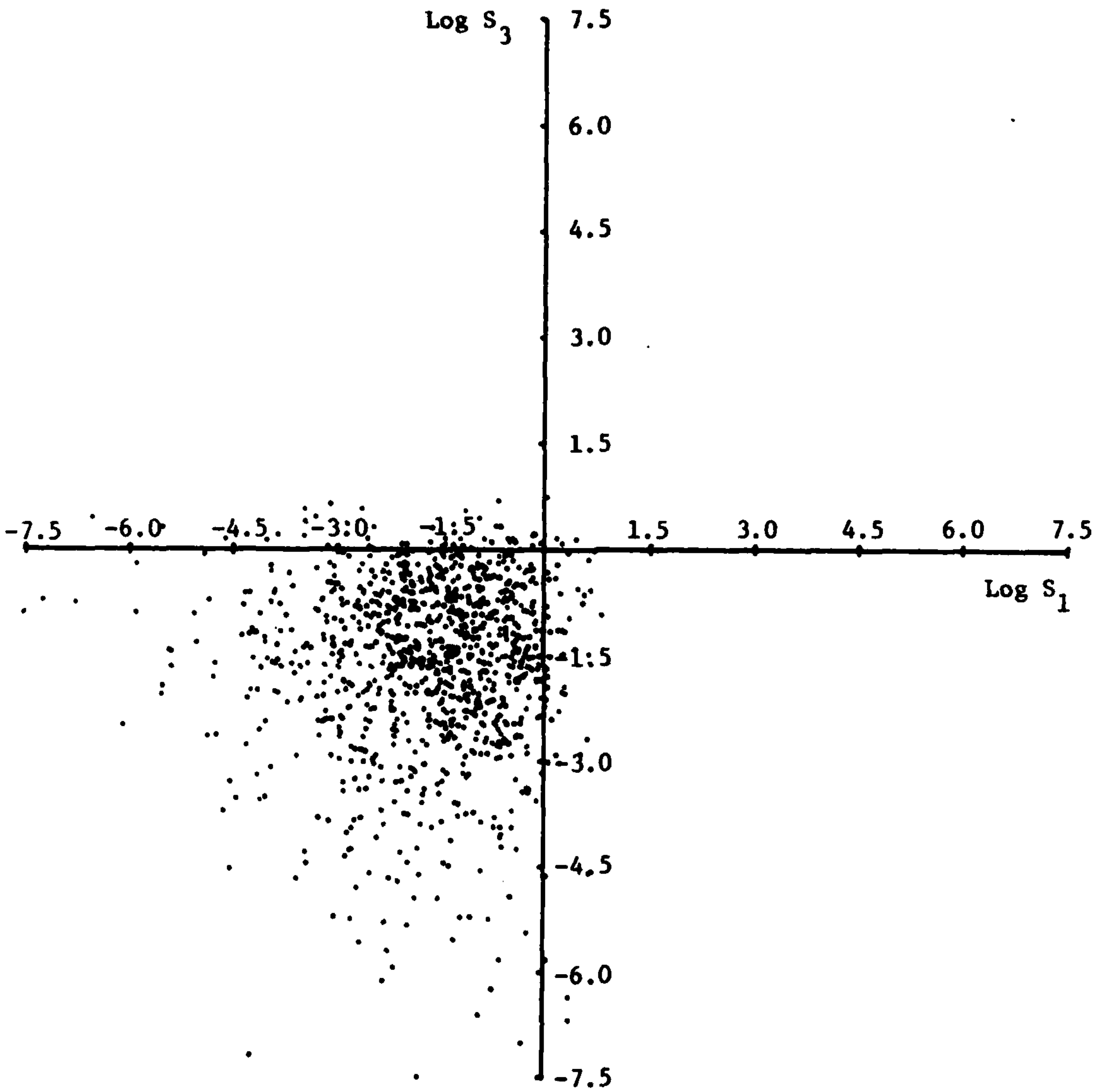

Figure 1.9. Scatter Plot of Log S_1 vs. Log S_3

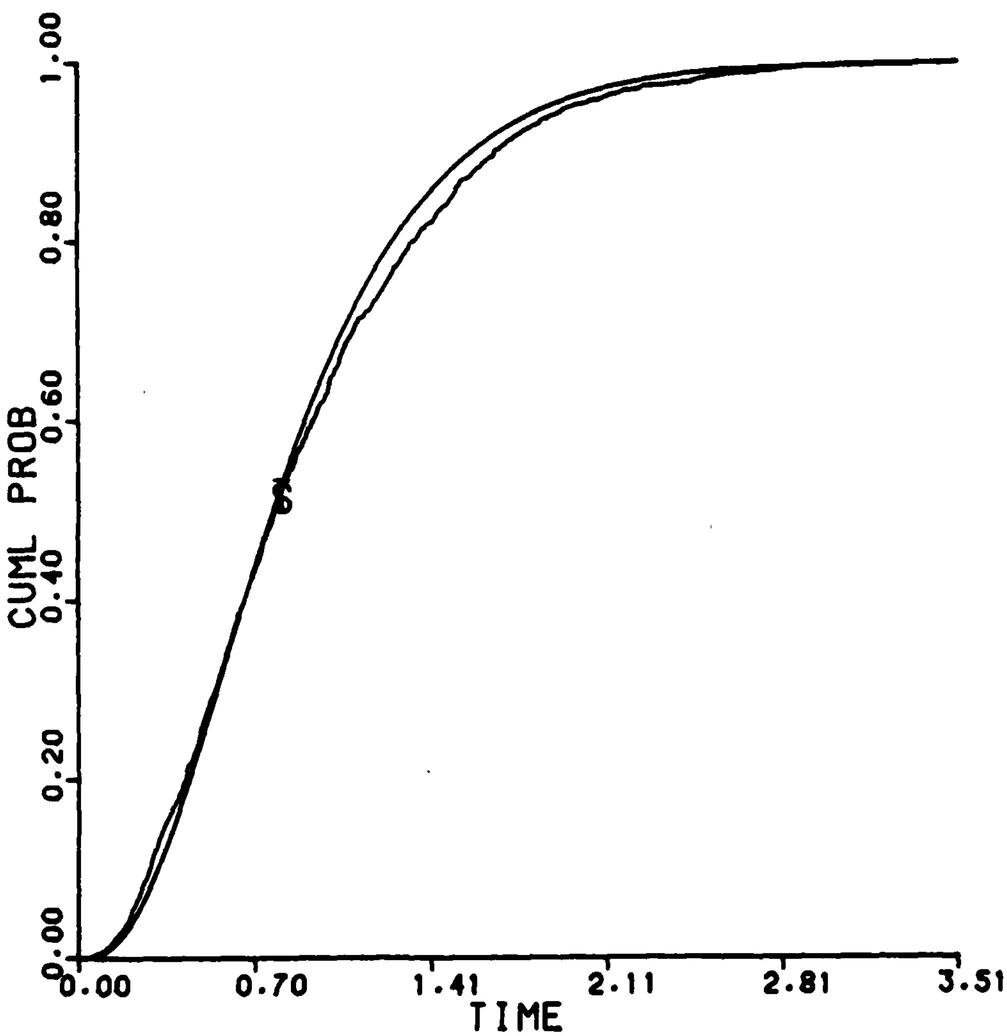

Figure 1.10. A Comparison Between a Sample Distribution, S, and one Assuming Independence, T

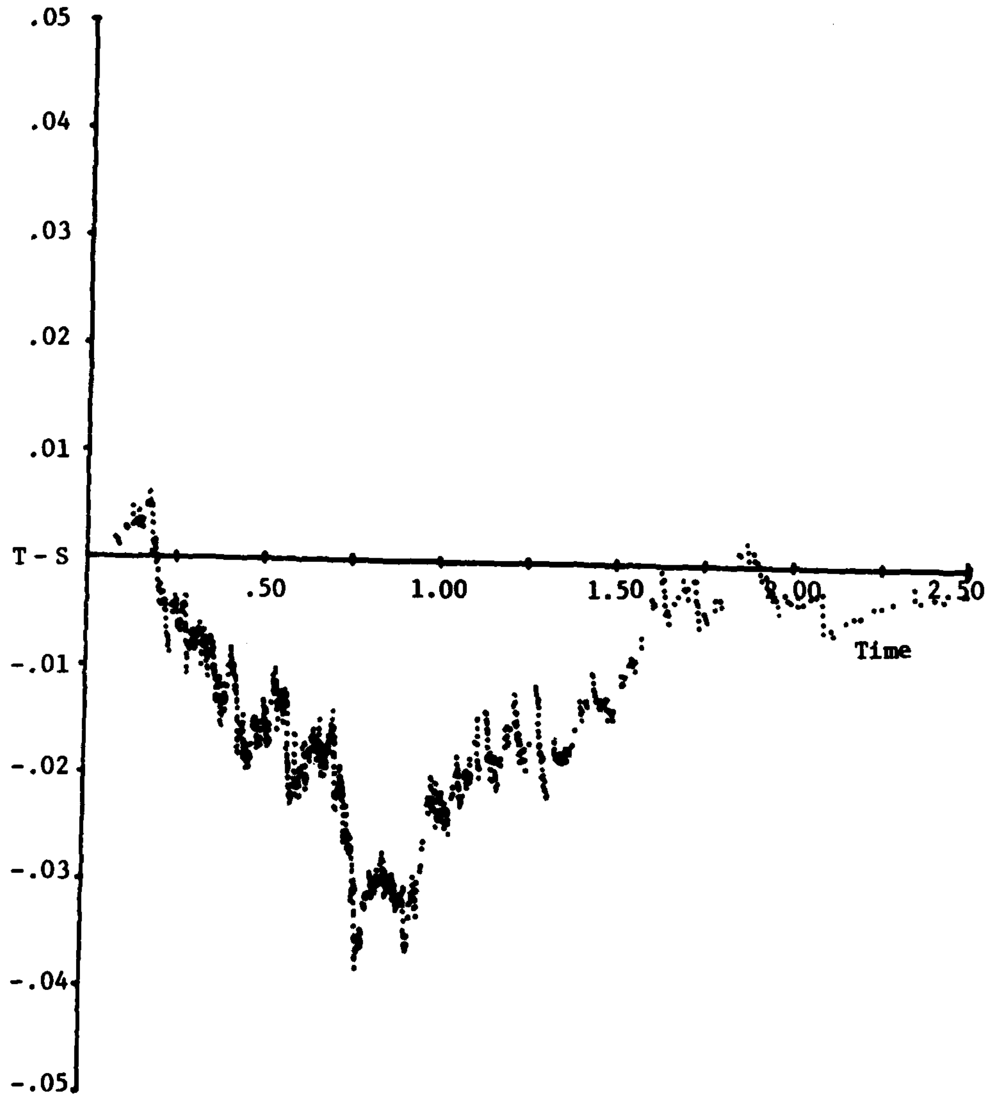

Figure 1.11. The Difference Between T and S

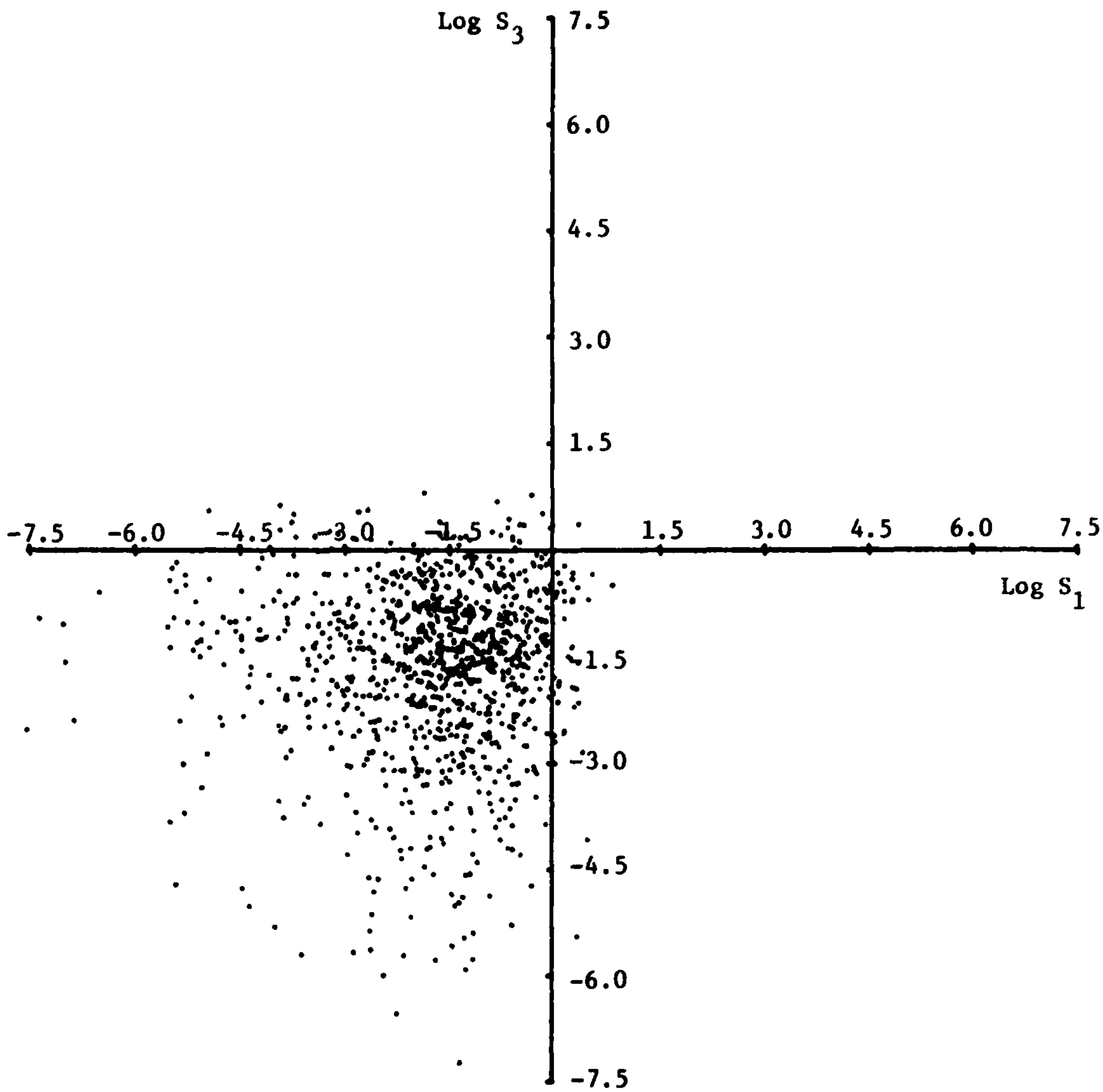

Figure 1.12. Scatter Plot of Log S_1 vs. Log S_3

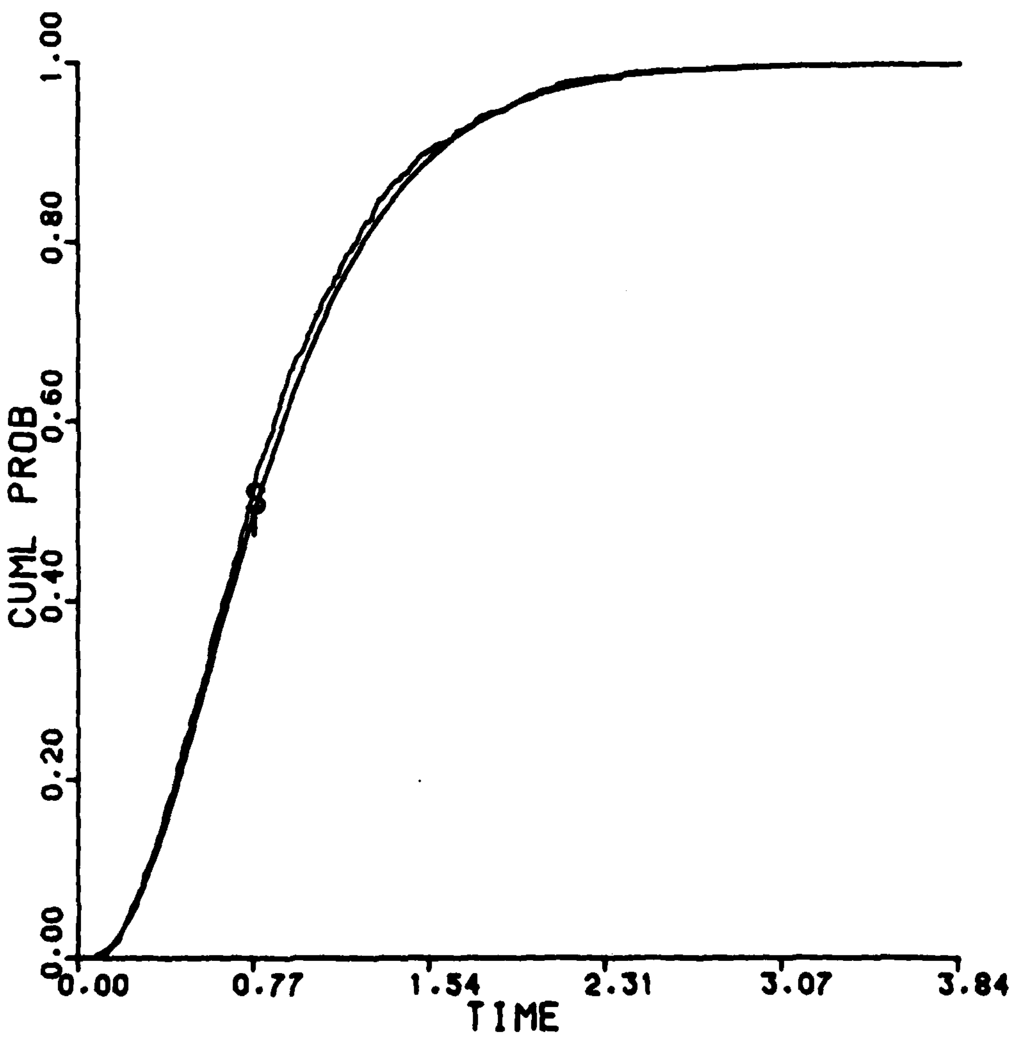

Figure 1.13. A Comparison Between a Sample Distribution, S, and one Assuming Independence, T

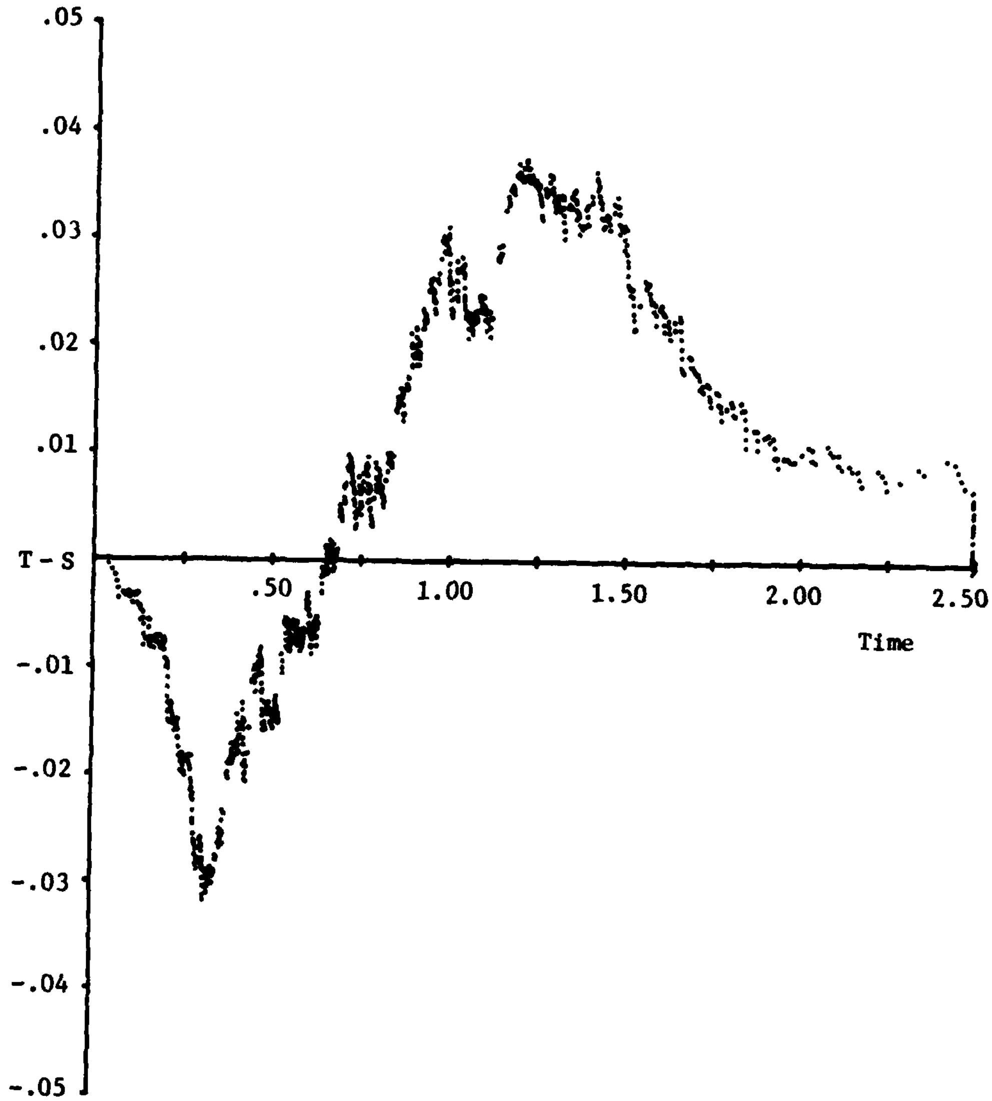

Figure 1.14. The Difference Between T and S

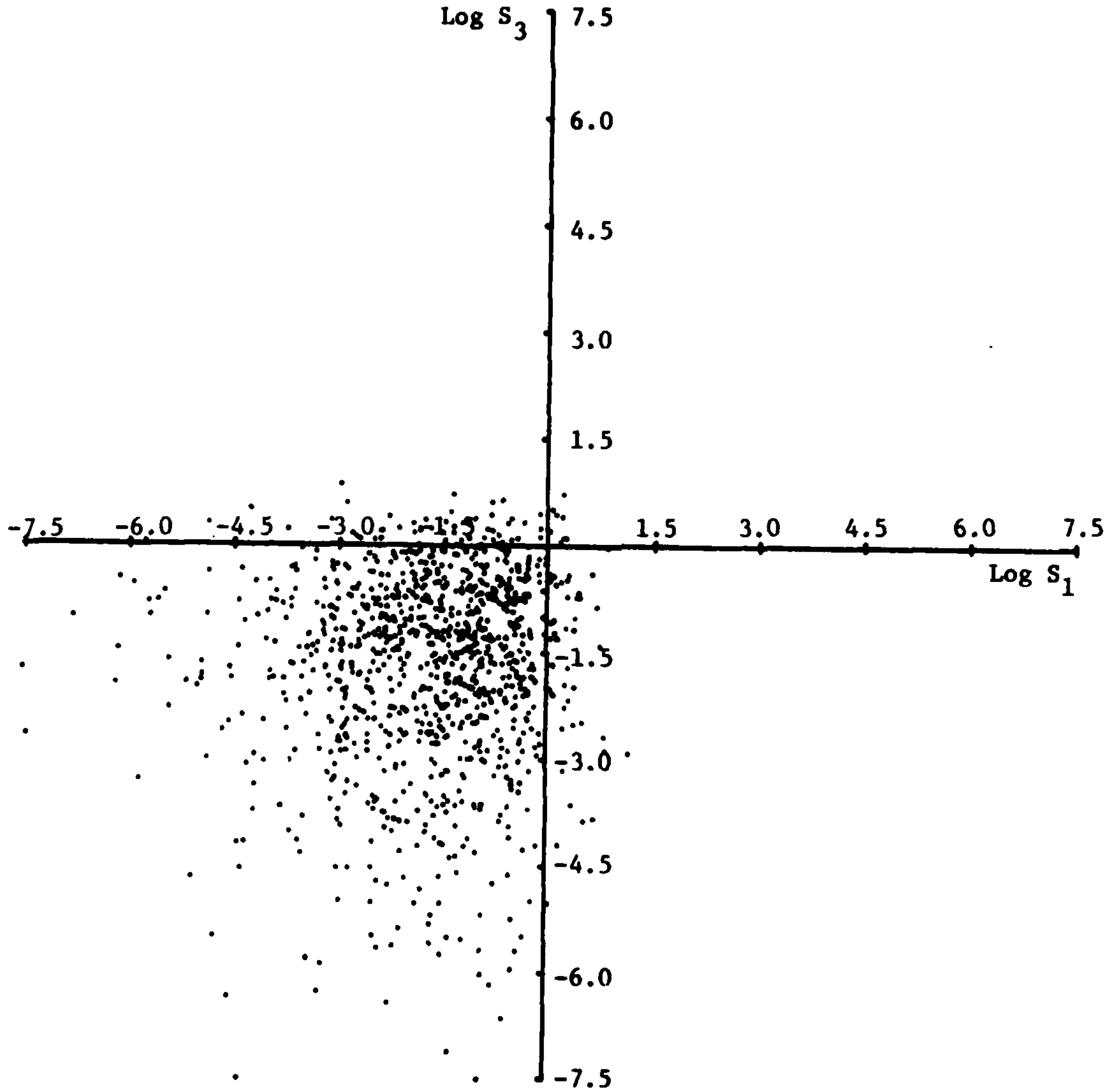

Figure 1.15. Scatter Plot of Log S_1 vs. Log S_3

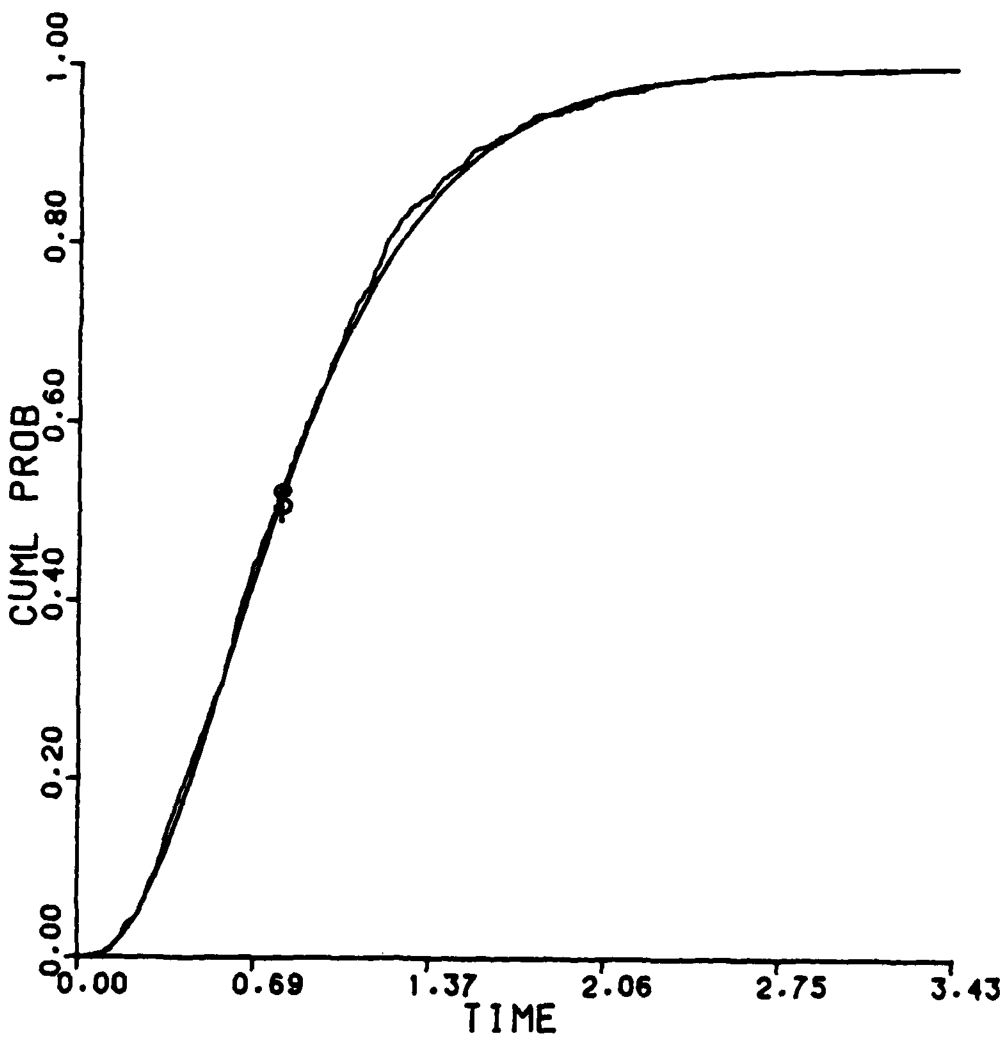

Figure 1.4. A Comparison Between a Sample Distribution, S, and one Assuming Independence, T

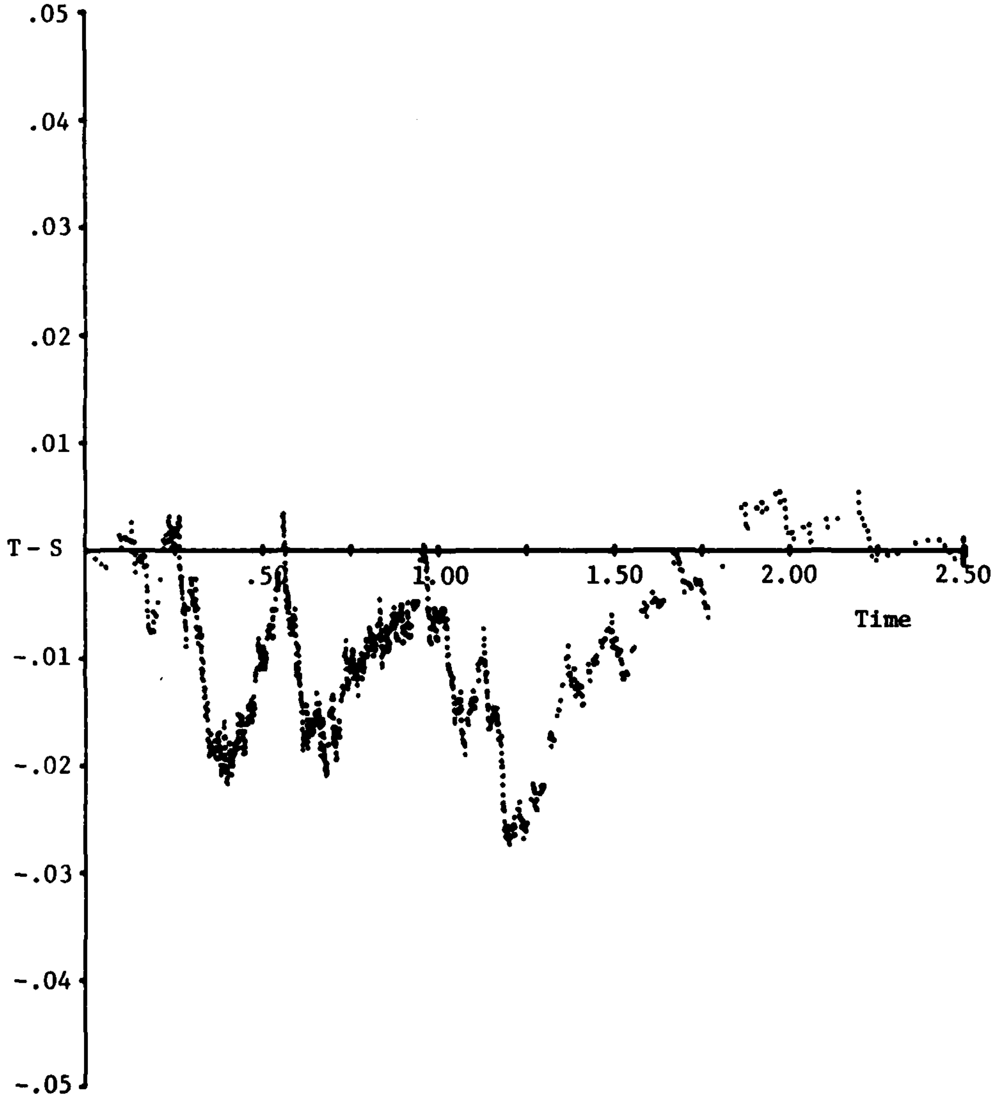

Figure 1.5. The Difference Between T and S

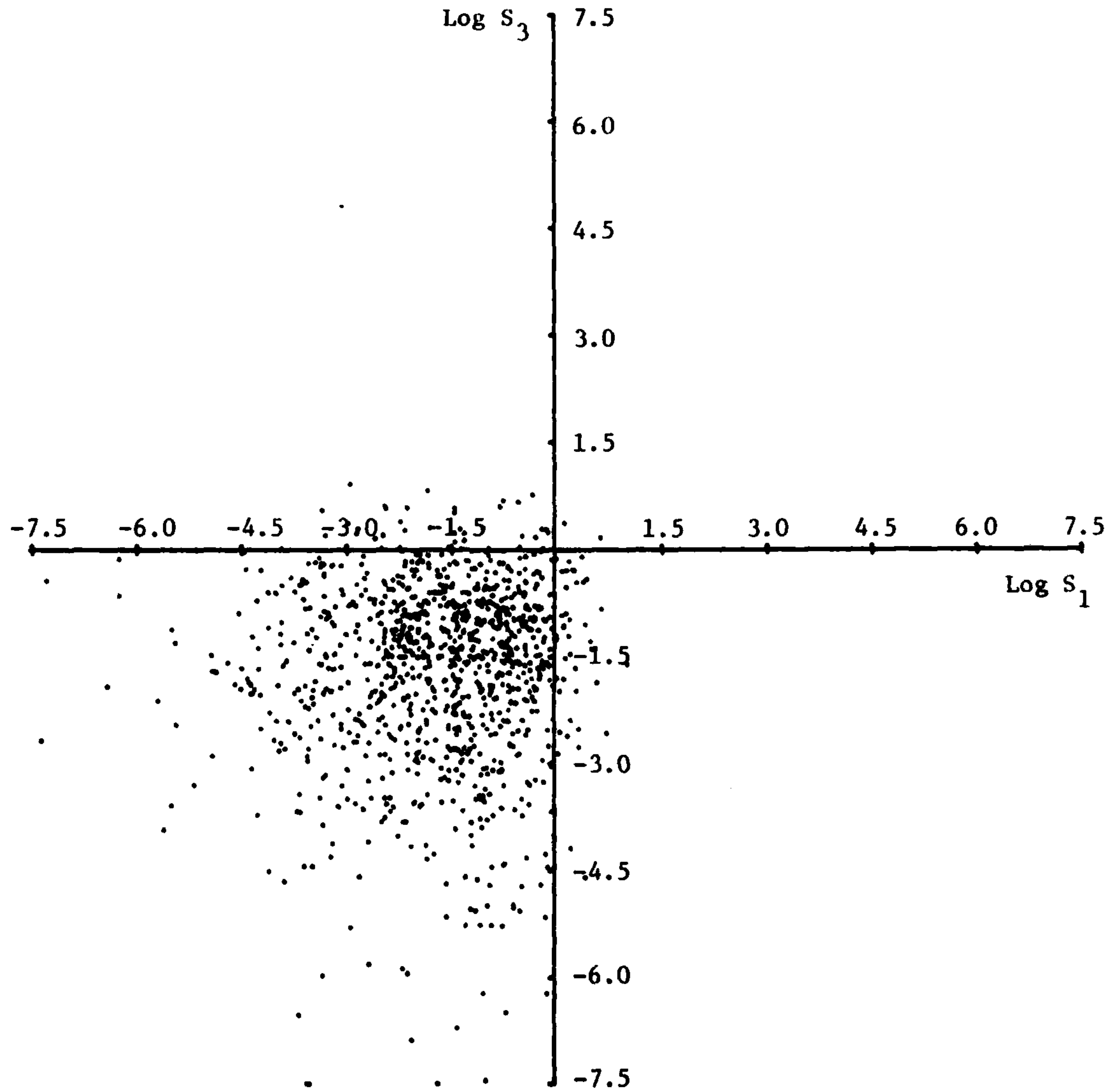

Figure 1.6. Scatter Plot of Log S_1 vs. Log S_3

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